Multi-Dimensional G–Brownian Motion and Related Stochastic Calculus under G–Expectation

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1st version: arXiv:math.PR/0601699 v1 28 Jan 2006
This version: November 9, 2006

Abstract. We develop a notion of nonlinear expectation —-G–expectation—- generated by a nonlinear heat equation with infinitesimal generator G. We first study multi-dimensional G–normal distributions. With this nonlinear distribution we can introduce our G–expectation under which the canonical process is a multi–dimensional G–Brownian motion. We then establish the related stochastic calculus, especially stochastic integrals of Itô's type with respect to our G–Brownian motion and derive the related Itô's formula. We have also obtained the existence and uniqueness of stochastic differential equation under our G–expectation.

Keywords: g–expectation, G–expectation, G–normal distribution, BSDE, SDE, nonlinear probability theory, nonlinear expectation, Brownian motion, Itô's stochastic calculus, Itô's integral, Itô's formula, Gaussian process, quadratic variation process, Jensen's inequality, G-convexity.

MSC 2000 Classification Numbers: 60H10, 60H05, 60H30, 35K55, 35K15, 49L25

*The author thanks the partial support from the Natural Science Foundation of China, grant No. 10131040. He thanks to the anonymous referee of SPA for his constructive suggestions. In particular, the discussions on Jensen's inequlity for G–convex functions in Section 6 were stimulated by one of his interesting questions.

1 Introduction

The purpose of this paper is to extend classical stochastic calculus for multidimensional Brownian motion to the setting of nonlinear G–expectation. We first recall the general framework of nonlinear expectation studied in [44] and [43], where the usual linearity is replaced by positive homogeneity and subadditivity. Such a sublinear expectation functional enables us to construct a Banach space, similar to an $\mathbb{L}^1$-space, starting from a functional lattice of Daniell's type.

Then we proceed to construct a sublinear expectation on the space of continuous paths from $\mathbb{R}_+$ to $\mathbb{R}^d$, starting from 0, which will be an analogue of Wiener's law. The operation mainly consists in replacing the Brownian semigroup by a nonlinear semigroup coming from the solution of a nonlinear parabolic partial differential equation (1) where appears a mapping G acting on Hessian matrices. Indeed, the Markov property permits to define in the same way nonlinear conditional expectations with respect to the past. Then we presents some rules and examples of computations under the newly constructed G-Brownian (motion) expectation. The fact that the underlying marginal nonlinear expectations are G-normal distributions derived from the nonlinear heat equation (1) is very helpful to estimate natural functionals. As result, our G–Brownian motion also has independent increments with identical G-normal distributions.

G–Brownian motion has a very rich and interesting new structure which non trivially generalizes the classical one. We thus can establish the related stochastic calculus, especially G–Itô's integrals (see [26, 1942]) and the related quadratic variation process $\langle B \rangle$. A very interesting new phenomenon of our G-Brownian motion is that its quadratic process $\langle B \rangle$ also has independent increments which are identically distributed. The corresponding G–Itô's formula is obtained. We then introduce the notion of G–martingales and the related Jensen inequality for a new type of "G–convex" functions. We have also established the existence and uniqueness of solution to stochastic differential equation under our stochastic calculus by the same Picard iterations as in the classical situation. Books on stochastic calculus e.g., [10], [23], [25], [27], [31], [36], [50], [51], [55] are recommended for understanding the present results and some further possible developments of this new stochastic calculus.

As indicated in Remark 2, the nonlinear expectations discussed in this paper can be regarded as coherent risk measures. This with the related conditional expectations $\mathbb{E}[\cdot|\mathcal{H}_t]_{t\geq 0}$ makes a dynamic risk measure: G–risk measure.

The other motivation of our G–expectation is the notion of (nonlinear) g–expectations introduced in [39], [40]. Here g is the generating function of a backward stochastic differential equation (BSDE) on a given probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The natural definition of the conditional g–expectations with respect to the past induces rich properties of nonlinear g–martingale theory (see among others, [3], [5], [6], [7], [11], [12], [8], [9], [28], [29], [41], [45], [46], [48]). Recently g–expectations are also studied as dynamic risk measures: g–risk measure (cf. [52], [4], [17]). Fully nonlinear super-hedging is also a possible application (cf. [33] and [53] where new BSDE approach was introduced).

The notion of g–expectation is defined on a given probability space. In [44]

(see also [43]), we have constructed a kind of filtration–consistent nonlinear expectations through the so–called nonlinear Markov chain. As compared with the framework of g–expectations, the theory of G–expectation is intrinsic, a meaning similar to "intrinsic geometry" in the sense that it is not based on a given (linear) probability space. Since the classical Brownian expectation as well as many other linear and nonlinear expectations are dominated by our G–Expectation (see Remark 25, Example 40 and [44]) and thus can be considered as continuous functionals, our theory also provides a flexible theoretical framework.

1–dimensional G–Brownian motion was studied in [47]. Unlike the classical situation, in general, we cannot find a system of coordinates under which the corresponding components B^i, $i = 1, \cdots, d$, are mutually independent from each others. The mutual quadratic variations $\langle B^i, B^j \rangle$ will play essentially important rule.

During the reversion process of this paper, the author has found a very interesting paper [18] by Denis and Martini on super-pricing of contingent claims under model uncertainty of volatility. They have introduced a norm on the space of continuous paths $\Omega = C([0, T])$ which corresponds to our L^2_G-norm and developed a stochastic integral. There is no notions of nonlinear expectation such as G–expectation, conditional G–expectation, the related G-normal distribution and the notion of independence in their paper. But on the other hand, powerful tools in capacity theory enables them to obtain pathwise results of random variables and stochastic processes through the language of "quasi-surely", (see Feyel and de La Pradelle [21]) in the place of "almost surely" in classical probability theory. Their method provides a way to proceed a pathwise analysis for our G–Brownian motion and the related stochastic calculus under G–expectation, see our forthcoming paper joint with Denis.

This paper is organized as follows: in Section 2, we recall the framework of nonlinear expectation established in [44] and adapt it to our objective. In section 3 we introduce d–dimensional G-normal distribution and discuss its main properties. In Section 4 we introduce d–dimensional G-Brownian motion, the corresponding G–expectation and their main properties. We then can establish stochastic integral with respect to G-Brownian motion of Itô's type, the related quadratic variation processes and then G–Itô's formula in Section 5, G–martingale and the Jensen's inequality for G–convex functions in Section 6 and the existence and uniqueness theorem of SDE driven by G-Brownian motion in Section 7.

The whole results of this paper are based on the very basic knowledge of Banach space and the parabolic partial differential equation (1). When this G-heat equation (1) is linear, our G-Brownian motion becomes the classical Brownian motion. This paper still provides an analytic shortcut to reach the sophistic Itô's calculus.

2 Nonlinear expectation: a general framework

We briefly recall the notion of nonlinear expectations introduced in [44]. Following Daniell's famous integration (cf. Daniell 1918 [14], see also [54]), we begin with a vector lattice. Let Ω be a given set and let $\mathcal{H}$ be a vector lattice of real functions defined on Ω containing 1, namely, $\mathcal{H}$ is a linear space such that $1 \in \mathcal{H}$ and that $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$. $\mathcal{H}$ is a space of random variables. We assume the functions on $\mathcal{H}$ are all bounded.

Definition 1 ***A nonlinear expectation*** *$\mathbb{E}$ is a functional $\mathcal{H} \mapsto \mathbb{R}$ satisfying the following properties*

(a) Monotonicity: *if $X, Y \in \mathcal{H}$ and $X \geq Y$ then $\mathbb{E}[X] \geq \mathbb{E}[Y]$.*
(b) Preserving of constants: *$\mathbb{E}[c] = c$.*

In this paper we are interested in the sublinear expectations which satisfy

(c) Sub-additivity (or self–dominated property):

$$\mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}[X - Y], \quad \forall X, Y \in \mathcal{H}.$$

(d) Positive homogeneity: *$\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \ \forall \lambda \geq 0, X \in \mathcal{H}$.*
(e) Constant translatability: *$\mathbb{E}[X + c] = \mathbb{E}[X] + c$.*

Remark 2 *It is clear that (d)+(e) implies (b). We recall the notion of the above sublinear expectations was systematically introduced by Artzner, Delbaen, Eber and Heath [1], [2], in the case where Ω is a finite set, and by Delbaen [16] in general situation with the notation of risk measure: $\rho(X) = \mathbb{E}[-X]$. See also in Huber [24] for even early study of this notion $\mathbb{E}$ (called upper expectation $\mathbf{E}^*$ in Ch.10 of [24]).*

We follow [44] to introduce a Banach space via $\mathcal{H}$ and $\mathbb{E}$. We denote $\|X\| := \mathbb{E}[|X|]$, $X \in \mathcal{H}$. $\mathcal{H}$ forms a normed space $(\mathcal{H}, \|\cdot\|)$ under $\|\cdot\|$ in the following sense. For each $X, Y \in \mathcal{H}$ such that $\|X - Y\| = 0$, we set $X = Y$. This is equivalent to say that the linear subspace

$$\mathcal{H}_0 := \{X \in \mathcal{H}, \ \|X\| = 0\}$$

is the null space, or in other words, we only consider the elements in the quotient space $\mathcal{H}/\mathcal{H}_0$. Under such arrangement $(\mathcal{H}, \|\cdot\|)$ is a normed space. We denote by $([\mathcal{H}], \|\cdot\|)$, or simply $[\mathcal{H}]$, the completion of $(\mathcal{H}, \|\cdot\|)$. $(\mathcal{H}, \|\cdot\|)$ is a dense subspace of the Banach space $([\mathcal{H}], \|\cdot\|)$ (see e.g., Yosida [56] Sec. I-10).

For any $X \in \mathcal{H}$, the mappings

$$X^+(\omega) = \max\{X(\omega), 0\} : \mathcal{H} \longmapsto \mathcal{H},$$
$$X^-(\omega) = \max\{-X(\omega), 0\} : \mathcal{H} \longmapsto \mathcal{H}$$

satisfy

$$|X^+ - Y^+| \leq |X - Y|,$$
$$X^- - Y^- \leq (Y - X)^+ \leq |X - Y|.$$

Thus they are both contract mappings under $\|\cdot\|$ and can be continuously extended to the Banach space $[\mathcal{H}]$.

We define the partial order "$\geq$" in this Banach space.

Definition 3 *An element X in $([\mathcal{H}], \|\cdot\|)$ is said to be nonnegative, or $X \geq 0$, $0 \leq X$, if $X = X^+$. We also denote by $X \geq Y$, or $Y \leq X$. if $X - Y \geq 0$.*

It is easy to check that if $X \geq Y$ and $Y \geq X$, then $X = Y$ in $([\mathcal{H}], \|\cdot\|)$. The nonlinear expectation $\mathbb{E}[\cdot]$ can be continuously extended to $([\mathcal{H}], \|\cdot\|)$ on which **(a)**–**(e)** still hold.

3 G–normal distributions

For a given positive integer n, we will denote by (x, y) the scalar product of x, $y \in \mathbb{R}^n$ and by $|x| = (x, x)^{1/2}$ the Euclidean norm of x. We denote by $lip(\mathbb{R}^n)$ the space of all bounded and Lipschitz real functions on $\mathbb{R}^n$. We introduce the notion of nonlinear distribution– G–normal distribution. A **G–normal distribution** is a nonlinear expectation defined on $lip(\mathbb{R}^d)$ (here $\mathbb{R}^d$ is considered as Ω and $lip(\mathbb{R}^d)$ as $\mathcal{H}$):

$$P_1^G(\phi) = u(1, 0) : \phi \in lip(\mathbb{R}^d) \mapsto \mathbb{R}$$

where $u = u(t, x)$ is a bounded continuous function on $[0, \infty) \times \mathbb{R}^d$ which is the viscosity solution of the following nonlinear parabolic partial differential equation (PDE)

$$\frac{\partial u}{\partial t} - G(D^2 u) = 0, \quad u(0, x) = \phi(x), \ (t, x) \in [0, \infty) \times \mathbb{R}^d, \tag{1}$$

here $D^2 u$ is the Hessian matrix of u, i.e., $D^2 u = (\partial^2_{x^i x^j} u)^d_{i,j=1}$ and

$$G(A) = G_\Gamma(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \mathrm{tr}[\gamma \gamma^T A], \quad A = (A_{ij})^d_{i,j=1} \in \mathbb{S}_d. \tag{2}$$

$\mathbb{S}_d$ denotes the space of $d \times d$ symmetric matrices. Γ is a given non empty, bounded and closed subset of $\mathbb{R}^{d \times d}$, the space of all $d \times d$ matrices.

Remark 4 *The nonlinear heat equation (1) is a special kind of Hamilton–Jacobi–Bellman equation. The existence and uniqueness of (1) in the sense of viscosity solution can be found in, for example, [13], [22], [38], [55], and [32] for $C^{1,2}$-solution if $\gamma\gamma^T \geq \sigma_0 I_n$, for each $\gamma \in \Gamma$, for a given constant $\sigma_0 > 0$ (see also in [36] for elliptic cases). It is a known result that $u(t, \cdot) \in lip(\mathbb{R}^d)$ (see e.g. [55] Ch.4, prop.3.1. or [38] Lemma 3.1. for the Lipschitz continuity of $u(t, \cdot)$,*

or Lemma 5.5 and Proposition 5.6 in [43] for a more general conclusion). The boundedness is simply from the comparison theorem (or maximum principle) of this PDE. It is also easy to check that, for a given $\psi \in lip(\mathbb{R}^d \times \mathbb{R}^d)$, $P_1^G(\psi(x,\cdot))$ is still a bounded and Lipschitz function in x.

In the case where Γ is a singleton $\{\gamma_0\}$, the above PDE becomes a standard linear heat equation and thus, for $G^0 = G_{\{\gamma_0\}}$, the corresponding G^0–distribution is just the d–dimensional classical normal distribution $\mathcal{N}(0, \gamma_0\gamma_0^T)$. In a typical case where $\gamma_0 = I_d \in \Gamma$, we have

$$P_1^{G^0}(\phi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp[-\sum_{i=1}^d \frac{(x^i)^2}{2}]\phi(x)dx.$$

In the case where $\gamma_0 \in \Gamma$, from comparison theorem of PDE,

$$P_1^G(\phi) \geq P_1^{G^0}(\phi), \ \forall \phi \in lip(\mathbb{R}^d). \tag{3}$$

More generally, for each subset $\Gamma' \subset \Gamma$, the corresponding $P^{G_{\Gamma'}}$–distribution is dominated by P^G in the following sense:

$$P_1^{G_{\Gamma'}}(\phi) - P_1^{G_{\Gamma'}}(\psi) \leq P_1^G(\phi - \psi), \quad \forall \phi, \psi \in lip(\mathbb{R}^d).$$

Remark 5 *In [47] we have discussed* 1*–dimensional case, which corresponds* $d = 1$ *and* $\Gamma = [\sigma, 1] \subset \mathbb{R}$*, where* $\sigma \in [0,1]$ *is a given constant. In this case the nonlinear heat equation (1) becomes*

$$\frac{\partial u}{\partial t} - \frac{1}{2}[(\partial_{xx}^2 u)^+ - \sigma^2(\partial_{xx}^2 u)^-] = 0, \quad u(0,x) = \phi(x), \ (t,x) \in [0,\infty) \times \mathbb{R}.$$

In multi–dimensional case we also have the following typical nonlinear heat equation:

$$\frac{\partial u}{\partial t} - \frac{1}{2}\sum_{i=1}^d [(\partial_{x^i x^i}^2 u)^+ - \sigma_i^2(\partial_{x^i x^i}^2 u)^-] = 0$$

where $\sigma_i \in [0,1]$ *are given constants. In this case we have*

$$\Gamma = \{diag[\gamma_1, \cdots, \gamma_d], \gamma_i \in [\sigma_i, 1], \ i = 1, \cdots, d\}.$$

The corresponding normal distribution with mean at $x \in \mathbb{R}^d$ and square variation $t > 0$ is $P_1^G(\phi(x + \sqrt{t} \times \cdot))$. Just like the classical situation of a normal distribution,, we have

Lemma 6 *For each* $\phi \in lip(\mathbb{R}^d)$*, the function*

$$u(t,x) = P_1^G(\phi(x + \sqrt{t} \times \cdot)), \quad (t,x) \in [0,\infty) \times \mathbb{R}^d \tag{4}$$

is the solution of the nonlinear heat equation (1) with the initial condition $u(0,\cdot) = \phi(\cdot)$*.*

Proof. Let $u \in C([0,\infty)\times\mathbb{R}^d)$ be the viscosity solution of (1) with $u(0,\cdot)=\phi(\cdot)\in lip(\mathbb{R}^d)$. For a fixed $(\bar{t},\bar{x})\in(0,\infty)\times\mathbb{R}^d$, we denote $\bar{u}(t,x)=u(t\times\bar{t},x\sqrt{\bar{t}}+\bar{x})$. Then $\bar{u}$ is the viscosity solution of (1) with the initial condition $\bar{u}(0,x)=\phi(x\sqrt{\bar{t}}+\bar{x})$. Indeed, let ψ be a $C^{1,2}$ function on $(0,\infty)\times\mathbb{R}^d$ such that $\psi\geq\bar{u}$ (resp. $\psi\leq\bar{u}$) and $\psi(\tau,\xi)=\bar{u}(\tau,\xi)$ for a fixed $(\tau,\xi)\in(0,\infty)\times\mathbb{R}^d$. We have $\psi(\frac{t}{\bar{t}},\frac{x-\bar{x}}{\sqrt{\bar{t}}})\geq u(t,x)$, for all (t,x) and

$$\psi(\frac{t}{\bar{t}},\frac{x-\bar{x}}{\sqrt{\bar{t}}})=u(t,x),\ \text{at}\ (t,x)=(\tau\bar{t},\xi\sqrt{\bar{t}}+\bar{x}).$$

Since u is the viscosity solution of (1), at the point $(t,x)=(\tau\bar{t},\xi\sqrt{\bar{t}}+\bar{x})$, we have

$$\frac{\partial\psi(\frac{t}{\bar{t}},\frac{x-\bar{x}}{\sqrt{\bar{t}}})}{\partial t}-G(D^2\psi(\frac{t}{\bar{t}},\frac{x-\bar{x}}{\sqrt{\bar{t}}}))\leq 0\quad(\text{resp.}\ \geq 0).$$

But G is a positive homogenous function, i.e., $G(\lambda A)=\lambda G(A)$, when $\lambda\geq 0$, we thus derive

$$\frac{\partial\psi(t,x)}{\partial t}-G(D^2\psi(t,x))|_{(t,x)=(\tau,\xi)}\leq 0\quad(\text{resp.}\ \geq 0).$$

This implies that $\bar{u}$ is the viscosity subsolution (resp. supersolution) of (1). According to the definition of $P^G(\cdot)$ we obtain (4). ■

Definition 7 *We denote*

$$P^G_t(\phi)(x)=P^G_1(\phi(x+\sqrt{t}\times\cdot))=u(t,x),\quad (t,x)\in[0,\infty)\times\mathbb{R}^d. \tag{5}$$

From the above lemma, for each $\phi\in lip(\mathbb{R}^d)$, we have the following nonlinear version of chain rule:

$$P^G_t(P^G_s(\phi))(x)=P^G_{t+s}(\phi)(x),\quad s,t\in[0,\infty),\ x\in\mathbb{R}^d. \tag{6}$$

This chain rule was initialled by Nisio [34] and [35] in terms of "envelope of Markovian semi-groups". See also [44].

Lemma 8 *The solution of (1) with initial condition $u(0,x)=\phi((\mathbf{a},x))$, for a given $\phi\in lip(\mathbb{R})$, has the form $u(t,x)=\bar{u}(t,\bar{x})$, $\bar{x}=(\mathbf{a},x)$, where $\bar{u}$ is the solution of*

$$\frac{\partial\bar{u}}{\partial t}-G_{\mathbf{a}}(\partial_{\bar{x}\bar{x}}\bar{u})=0,\quad u(0,\bar{x})=\phi(\bar{x}),\ (t,\bar{x})\in[0,\infty)\times\mathbb{R}, \tag{7}$$

where

$$G_{\mathbf{a}}(\beta)=\frac{1}{2}\max_{\gamma\in\Gamma}tr[\gamma\gamma^T\mathbf{a}\mathbf{a}^T\beta],\quad\beta\in\mathbb{R}.$$

The above PDE can be written

$$\frac{\partial\bar{u}}{\partial t}-\frac{1}{2}[\sigma_{\mathbf{a}\mathbf{a}^T}(\partial_{\bar{x}\bar{x}}\bar{u})^+ +\sigma_{-\mathbf{a}\mathbf{a}^T}(\partial_{\bar{x}\bar{x}}\bar{u})^-]=0,\quad u(0,\bar{x})=\phi(\bar{x}). \tag{8}$$

where we denote $\mathbf{a}\mathbf{a}^T = [a^i a^j]_{i,j=1}^d \in \mathbb{S}_d$ *and*

$$\sigma_A = \sup_{\gamma\in\Gamma} tr[\gamma\gamma^T A] = 2G(A), \quad A \in \mathbb{S}_d. \tag{9}$$

Here $\mathbb{S}_d$ *is the space of* $d\times d$ *symmetric matrices.*

Remark 9 *It is clear that the functional*

$$P_1^{G_{\mathbf{a}}}(\phi) = \bar{u}(1,0) : \phi \in lip(\mathbb{R}) \mapsto \mathbb{R}$$

constitutes a special 1*–dimensional nonlinear normal distribution, called* $G_{\mathbf{a}}$*–* ***normal distribution.***

Proof. It is clear that the PDE (7) has a unique viscosity solution. We then can set $u(t,x) = \bar{u}(t,(\mathbf{a},x))$ and check that u is the viscosity solution of (1). (8) is then easy to check. ■

Example 10 *In the above lemma, if* ϕ *is convex, and* $\sigma_{\mathbf{a}\mathbf{a}^T} > 0$*, then*

$$P_t^G(\phi((\mathbf{a},\cdot))(x) = \frac{1}{\sqrt{2\pi\sigma_{\mathbf{a}\mathbf{a}^T}t}}\int_{-\infty}^{\infty}\phi(y)\exp(-\frac{(y-x)^2}{2\sigma_{\mathbf{a}\mathbf{a}^T}t})dy.$$

If ϕ *is concave and* $\sigma_{-\mathbf{a}\mathbf{a}^T} < 0$*, then*

$$P_t^G(\phi((\mathbf{a},\cdot))(x) = \frac{1}{\sqrt{2\pi|\sigma_{-\mathbf{a}\mathbf{a}^T}|t}}\int_{-\infty}^{\infty}\phi(y)\exp(-\frac{(y-x)^2}{2|\sigma_{-\mathbf{a}\mathbf{a}^T}|t})dy.$$

Proposition 11 *We have*
(i) For each $t > 0$*, the* G*–normal distribution* P_t^G *is a nonlinear expectation on the lattice* $lip(\mathbb{R}^d)$*, with* $\Omega = \mathbb{R}^d$*, satisfying (a)–(e) of definition 1. The corresponding completion space* $[\mathcal{H}] = [lip(\mathbb{R}^d)]_t$ *under the norm* $\|\phi\|_t := P_t^G(|\phi|)(0)$ *contains* $\phi(x) = x_1^{n_1}\times\cdots\times x_d^{n_d}$*,* $n_i = 1,2,\cdots$*,* $i = 1,\cdots,d$*,* $x = (x_1,\cdots,x_d)^T$ *as well as* $x_1^{n_1}\times\cdots\times x_d^{n_d}\times\psi(x)$*,* $\psi\in lip(\mathbb{R}^d)$ *as its special elements. Relation (5) still holds. We also have the following properties*
(ii) We have, for each $\mathbf{a} = (a_1,\cdots,a_d)^T \in \mathbb{R}^d$ *and* $A \in \mathbb{S}_d$

$$\begin{aligned}
P_t^G((\mathbf{a},x)_{x\in\mathbb{R}^d}) &= 0,\\
P_t^G(((\mathbf{a},x)^2)_{x\in\mathbb{R}^d}) &= t\cdot\sigma_{\mathbf{a}\mathbf{a}^T}, \quad P_t^G((-(\mathbf{a},x)^2)_{x\in\mathbb{R}^d}) = t\cdot\sigma_{-\mathbf{a}\mathbf{a}^T},\\
P_t^G(((\mathbf{a},x)^4)_{x\in\mathbb{R}^d}) &= 6(\sigma_{\mathbf{a}\mathbf{a}^T})t^2, \quad P_t^G((-(\mathbf{a},x)^4)_{x\in\mathbb{R}^d}) = -6(\sigma_{-\mathbf{a}\mathbf{a}^T})^2t^2,\\
P_t^G(((Ax,x))_{x\in\mathbb{R}^d}) &= t\cdot\sigma_A = 2G(A)t.
\end{aligned}$$

Proof. (ii) By Lemma 8, we have the explicit solutions of the nonlinear PDE (1) with the following different initial condition $u(0,x) = \phi(x)$:

$$\begin{array}{rcl}
\phi(x) = (\mathbf{a},x) & \Longrightarrow & u(t,x) = (\mathbf{a},x),\\
\phi(x) = (\mathbf{a},x)^4 & \Longrightarrow & u(t,x) = (\mathbf{a},x)^4 + 6(\mathbf{a},x)^2\sigma_{\mathbf{a}\mathbf{a}^T}t + 6\sigma_{\mathbf{a}\mathbf{a}^T}^2t^2,\\
\phi(x) = -(\mathbf{a},x)^4 & \Longrightarrow & u(t,x) = -(\mathbf{a},x)^4 + 6(\mathbf{a},x)^2\sigma_{-\mathbf{a}\mathbf{a}^T}t - 6|\sigma_{-\mathbf{a}\mathbf{a}^T}|^2t^2.
\end{array}$$

Similarly, we can check that $\phi(x)=(Ax,x) \implies u(t,x)=(Ax,x)+\sigma_A t$. This implies, by setting $A=\mathbf{a}\mathbf{a}^T$ and $A=-\mathbf{a}\mathbf{a}^T$,

$$\begin{aligned}\phi(x)=(\mathbf{a},x)^2 &\implies u(t,x)=(\mathbf{a},x)^2+\sigma_{\mathbf{a}\mathbf{a}^T}t,\\ \phi(x)=-(\mathbf{a},x)^2 &\implies u(t,x)=-(\mathbf{a},x)^2+\sigma_{-\mathbf{a}\mathbf{a}^T}t.\end{aligned}$$

More generally, for $\phi(x)=(\mathbf{a},x)^{2n}$, we have

$$u(t,x)=\frac{1}{\sqrt{2\pi\sigma_{\mathbf{a}\mathbf{a}^T}t}}\int_{-\infty}^{\infty}y^{2n}\exp(-\frac{(y-x)^2}{2\sigma_{\mathbf{a}\mathbf{a}^T}t})dy.$$

By this we can prove (i). ■

4 G–Brownian motions under G–expectations

In the rest of this paper, we set $\Omega=C_0^d(\mathbb{R}^+)$ the space of all $\mathbb{R}^d$–valued continuous paths $(\omega_t)_{t\in\mathbb{R}^+}$, with $\omega_0=0$, equipped with the distance

$$\rho(\omega^1,\omega^2):=\sum_{i=1}^{\infty}2^{-i}[(\max_{t\in[0,i]}|\omega_t^1-\omega_t^2|)\wedge 1].$$

$\mathbf{\Omega}$ is the classical canonical space and $\omega=(\omega_t)_{t\geq 0}$ is the corresponding canonical process. It is well–known that in this canonical space there exists a Wiener measure $(\mathbf{\Omega},\mathcal{F},P)$ under which the canonical process $B_t(\omega)=\omega_t$ is a d–dimensional Brownian motion.

For each fixed $T\geq 0$, we consider the following space of random variables:

$$L_{ip}^0(\mathcal{H}_T):=\{X(\omega)=\phi(\omega_{t_1},\cdots,\omega_{t_m}),\forall m\geq 1,\ t_1,\cdots,t_m\in[0,T],\phi\in l_{ip}(\mathbb{R}^{d\times m})\}.$$

It is clear that $\{L_{ip}^0(\mathcal{H}_t)\}_{t\geq 0}$ constitute a family of sub-lattices such that $L_{ip}^0(\mathcal{H}_t)\subseteq L_{ip}^0(\mathcal{H}_T)$, for $t\leq T<\infty$. $L_{ip}^0(\mathcal{H}_t)$ representing the past history of ω at the time t. It's completion will play the same role of Brownian filtration $\mathcal{F}_t^B$ as in classical stochastic analysis. We also denote

$$L_{ip}^0(\mathcal{H}):=\bigcup_{n=1}^{\infty}L_{ip}^0(\mathcal{H}_n).$$

Remark 12 *It is clear that $lip(\mathbb{R}^{d\times m})$ and then $L_{ip}^0(\mathcal{H}_T)$, $L_{ip}^0(\mathcal{H})$ are vector lattices. Moreover, since $\phi,\psi\in lip(\mathbb{R}^{d\times m})$ implies $\phi\cdot\psi\in lip(\mathbb{R}^{d\times m})$ thus X, $Y\in L_{ip}^0(\mathcal{H}_T)$ implies $X\cdot Y\in L_{ip}^0(\mathcal{H}_T)$; X, $Y\in L_{ip}^0(\mathcal{H})$ implies $X\cdot Y\in L_{ip}^0(\mathcal{H})$.*

We will consider the canonical space and set $B_t(\omega)=\omega_t$, $t\in[0,\infty)$, for $\omega\in\Omega$.

Definition 13 *The canonical process B is called a (d–dimensional) G–**Brownian motion** under a nonlinear expectation $\mathbb{E}$ defined on $L_{ip}^0(\mathcal{H})$ if*

(i) For each $s, t \geq 0$ and $\psi \in lip(\mathbb{R}^d)$, B_t and $B_{t+s} - B_s$ are identically distributed:

$$\mathbb{E}[\psi(B_{t+s} - B_s)] = \mathbb{E}[\psi(B_t)] = P_t^G(\psi).$$

(ii) For each $m = 1, 2, \cdots$, $0 \leq t_1 < \cdots < t_m < \infty$, the increment $B_{t_m} - B_{t_{m-1}}$ is "backwardly" independent from $B_{t_1}, \cdots, B_{t_{m-1}}$ in the following sense: for each $\phi \in lip(\mathbb{R}^{d \times m})$,

$$\mathbb{E}[\phi(B_{t_1}, \cdots, B_{t_{m-1}}, B_{t_m})] = \mathbb{E}[\phi_1(B_{t_1}, \cdots, B_{t_{m-1}})],$$

where $\phi_1(x^1, \cdots, x^{m-1}) = \mathbb{E}[\phi(x^1, \cdots, x^{m-1}, B_{t_m} - B_{t_{m-1}} + x^{m-1})]$, $x^1, \cdots, x^{m-1} \in \mathbb{R}^d$.
The related conditional expectation of $\phi(B_{t_1}, \cdots, B_{t_m})$ under $\mathcal{H}_{t_k}$ is defined by

$$\mathbb{E}[\phi(B_{t_1}, \cdots, B_{t_k}, \cdots, B_{t_m}) | \mathcal{H}_{t_k}] = \phi_{m-k}(B_{t_1}, \cdots, B_{t_k}), \tag{10}$$

where

$$\phi_{m-k}(x^1, \cdots, x^k) = \mathbb{E}[\phi(x^1, \cdots, x^k, B_{t_{k+1}} - B_{t_k} + x^k, \cdots, B_{t_m} - B_{t_k} + x^k)].$$

It is proved in [44] that $\mathbb{E}[\cdot]$ consistently defines a nonlinear expectation on the vector lattice $L^0_{ip}(\mathcal{H}_T)$ as well as on $L^0_{ip}(\mathcal{H})$ satisfying (a)–(e) in Definition 1. It follows that $\mathbb{E}[|X|]$, $X \in L^0_{ip}(\mathcal{H}_T)$ (resp. $L^0_{ip}(\mathcal{H})$) forms a norm and thus $L^0_{ip}(\mathcal{H}_T)$ (resp. $L^0_{ip}(\mathcal{H})$) can be extended, under this norm, to a Banach space. We denote this space by $L^1_G(\mathcal{H}_T)$ (resp. $L^1_G(\mathcal{H})$). For each $0 \leq t \leq T < \infty$, we have $L^1_G(\mathcal{H}_t) \subseteq L^1_G(\mathcal{H}_T) \subset L^1_G(\mathcal{H})$. In $L^1_G(\mathcal{H}_T)$ (resp. $L^1_G(\mathcal{H}_T)$), $\mathbb{E}[\cdot]$ still satisfies (a)–(e) in Definition 1.

Remark 14 *It is suggestive to denote $L^0_{ip}(\mathcal{H}_t)$ by $\mathcal{H}^0_t$ and $L^1_G(\mathcal{H}_t)$ by $\mathcal{H}_t$, $L^1_G(\mathcal{H})$ by $\mathcal{H}$ and thus consider the conditional expectation $\mathbb{E}[\cdot|\mathcal{H}_t]$ as a projective mapping from $\mathcal{H}$ to $\mathcal{H}_t$. The notation $L^1_G(\mathcal{H}_t)$ is due to the similarity of $L^1(\Omega, \mathcal{F}_t, P)$ in classical stochastic analysis.*

Definition 15 *The expectation $\mathbb{E}[\cdot] : L^1_G(\mathcal{H}) \mapsto \mathbb{R}$ introduced through above procedure is called G–**expectation**, or G–Brownian expectation. The corresponding canonical process B is said to be a G–Brownian motion under $\mathbb{E}[\cdot]$.*

For a given $p > 1$, we also denote $L^p_G(\mathcal{H}) = \{X \in L^1_G(\mathcal{H}),\ |X|^p \in L^1_G(\mathcal{H})\}$. $L^p_G(\mathcal{H})$ is also a Banach space under the norm $\|X\|_p := (\mathbb{E}[|X|^p])^{1/p}$. We have (see Appendix)

$$\|X + Y\|_p \leq \|X\|_p + \|Y\|_p$$

and, for each $X \in L^p_G$, $Y \in L^q_G(Q)$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|XY\| = \mathbb{E}[|XY|] \leq \|X\|_p \|X\|_q .$$

With this we have $\|X\|_p \leq \|X\|_{p'}$ if $p \leq p'$.

We now consider the conditional expectation introduced in (10). For each fixed $t = t_k \leq T$, the conditional expectation $\mathbb{E}[\cdot|\mathcal{H}_t] : L^0_{ip}(\mathcal{H}_T) \mapsto L^0_{ip}(\mathcal{H}_t)$ is

a continuous mapping under $\|\cdot\|$. Indeed, we have $\mathbb{E}[\mathbb{E}[X|\mathcal{H}_t]] = \mathbb{E}[X]$, $X \in L^0_{ip}(\mathcal{H}_T)$ and, since P^G_t is subadditive,

$$\mathbb{E}[X|\mathcal{H}_t] - \mathbb{E}[Y|\mathcal{H}_t] \leq \mathbb{E}[X-Y|\mathcal{H}_t] \leq \mathbb{E}[|X-Y||\mathcal{H}_t]$$

We thus obtain

$$\mathbb{E}[\mathbb{E}[X|\mathcal{H}_t] - \mathbb{E}[Y|\mathcal{H}_t]] \leq \mathbb{E}[X-Y]$$

and

$$\|\mathbb{E}[X|\mathcal{H}_t] - \mathbb{E}[Y|\mathcal{H}_t]\| \leq \|X-Y\|\,.$$

It follows that $\mathbb{E}[\cdot|\mathcal{H}_t]$ can be also extended as a continuous mapping $L^1_G(\mathcal{H}_T) \mapsto L^1_G(\mathcal{H}_t)$. If the above T is not fixed, then we can obtain $\mathbb{E}[\cdot|\mathcal{H}_t] : L^1_G(\mathcal{H}) \mapsto L^1_G(\mathcal{H}_t)$.

Proposition 16 *We list the properties of $\mathbb{E}[\cdot|\mathcal{H}_t]$, $t \in [0,T]$, that hold in $L^0_{ip}(\mathcal{H}_T)$ and still hold for X, $Y \in L^1_G(\mathcal{H}_T)$:*

(i) $\mathbb{E}[X|\mathcal{H}_t] = X$, *for* $X \in L^1_G(\mathcal{H}_t)$, $t \leq T$.
(ii) *If* $X \geq Y$, *then* $\mathbb{E}[X|\mathcal{H}_t] \geq \mathbb{E}[Y|\mathcal{H}_t]$.
(iii) $\mathbb{E}[X|\mathcal{H}_t] - \mathbb{E}[Y|\mathcal{H}_t] \leq \mathbb{E}[X-Y|\mathcal{H}_t]$.
(iv) $\mathbb{E}[\mathbb{E}[X|\mathcal{H}_t]|\mathcal{H}_s] = \mathbb{E}[X|\mathcal{H}_{t\wedge s}]$, $\mathbb{E}[\mathbb{E}[X|\mathcal{H}_t]] = \mathbb{E}[X]$.
(v) $\mathbb{E}[X+\eta|\mathcal{H}_t] = \mathbb{E}[X|\mathcal{H}_t] + \eta$, $\eta \in L^1_G(\mathcal{H}_t)$
(vi) $\mathbb{E}[\eta X|\mathcal{H}_t] = \eta^+\mathbb{E}[X|\mathcal{H}_t] + \eta^-\mathbb{E}[-X|\mathcal{H}_t]$, *for bounded* $\eta \in L^1_G(\mathcal{H}_t)$.
(vii) *We have the following independence:*

$$\mathbb{E}[X|\mathcal{H}_t] = \mathbb{E}[X], \quad \forall X \in L^1_G(\mathcal{H}^t_T),\ \forall T \geq 0,$$

where $L^1_G(\mathcal{H}^t_T)$ is the extension, under $\|\cdot\|$, of $L^0_{ip}(\mathcal{H}^t_T)$ which consists of random variables of the form $\phi(B^t_{t_1}, B^t_{t_2}, \cdots, B^t_{t_m})$, $\phi \in lip(\mathbb{R}^m)$, $t_1, \cdots, t_m \in [0,T]$, $m = 1, 2, \cdots$. Here we denote

$$B^t_s = B_{t+s} - B_t, \quad s \geq 0.$$

(viii) *The increments of B are identically distributed:*

$$\mathbb{E}[\phi(B^t_{t_1}, B^t_{t_2}, \cdots, B^t_{t_m})] = \mathbb{E}[\phi(B_{t_1}, B_{t_2}, \cdots, B_{t_m})].$$

The meaning of the independence in (vii) is similar to the classical one:

Definition 17 *An $\mathbb{R}^n$ valued random variable $Y \in (L^1_G(\mathcal{H}))^n$ is said to be independent of $\mathcal{H}_t$ for some given t if for each $\phi \in lip(\mathbb{R}^n)$ we have*

$$\mathbb{E}[\phi(Y)|\mathcal{H}_t] = \mathbb{E}[\phi(Y)].$$

It is seen that the above property (vii) also holds for the situation $X \in L^1_G(\mathcal{H}^t)$ where $L^1_G(\mathcal{H}^t)$ is the completion of the sub-lattice $\cup_{T\geq 0} L^1_G(\mathcal{H}^t_T)$ under $\|\cdot\|$.

From the above results we have

Proposition 18 *For each fixed $t \geq 0$, $(B_s^t)_{s\geq 0}$ is a G–Brownian motion in $L_G^1(\mathcal{H}^t)$ under the same G–expectation $\mathbb{E}[\cdot]$.*

Remark 19 *We can also prove, using Lemma 6, that the time scaling of B, i.e., $\tilde{B} = (\sqrt{\lambda}B_{t/\lambda})_{t\geq 0}$ also consists a G–Brownian motion.*

The following property is very useful

Proposition 20 *Let $X, Y \in L_G^1(\mathcal{H})$ be such that $\mathbb{E}[Y|\mathcal{H}_t] = -\mathbb{E}[-Y|\mathcal{H}_t]$, for some $t \in [0,T]$. Then we have*

$$\mathbb{E}[X+Y|\mathcal{H}_t] = \mathbb{E}[X|\mathcal{H}_t] + \mathbb{E}[Y|\mathcal{H}_t].$$

In particular, if $\mathbb{E}[Y|\mathcal{H}_t] = \mathbb{E}[-Y|\mathcal{H}_t] = 0$, then $\mathbb{E}[X+Y|\mathcal{H}_t] = \mathbb{E}[X|\mathcal{H}_t]$.

Proof. It is simply because we have $\mathbb{E}[X+Y|\mathcal{H}_t] \leq \mathbb{E}[X|\mathcal{H}_t] + \mathbb{E}[Y|\mathcal{H}_t]$ and

$$\mathbb{E}[X+Y|\mathcal{H}_t] \geq \mathbb{E}[X|\mathcal{H}_t] - \mathbb{E}[-Y|\mathcal{H}_t] = \mathbb{E}[X|\mathcal{H}_t] + \mathbb{E}[Y|\mathcal{H}_t].$$

■

Example 21 *From the last relation of Proposition 11-(ii), we have*

$$\mathbb{E}[(AB_t, B_t)] = \sigma_A t = 2G(A)t, \quad \forall A \in \mathbb{S}_d.$$

More general, for each $s \leq t$ and $\eta = (\eta^{ij})_{i,j=1}^d \in L_G^2(\mathcal{H}_s; \mathbb{S}_d)$,

$$\mathbb{E}[(\eta B_t^s, B_t^s)|\mathcal{H}_s] = \sigma_\eta t = 2G(\eta)t, \;\; s,t \geq 0. \tag{11}$$

Definition 22 *We will denote, in the rest of this paper,*

$$B_t^{\mathbf{a}} = (\mathbf{a}, B_t), \quad \textit{for each } \mathbf{a} = (a_1, \cdots, a_d)^T \in \mathbb{R}^d \tag{12}$$

From Lemma 8 and Remark 9,

$$\mathbb{E}[\phi(B_t^{\mathbf{a}})] = P_t^G(\phi((\mathbf{a}, \cdot))) = P_t^{G_{\mathbf{a}}}(\phi)$$

*where $P^{G_{\mathbf{a}}}$ is the (1–dimensional) $G_{\mathbf{a}}$–normal distribution. Thus, according to Definition 13 for d–dimensional G–Brownian motion, $B^{\mathbf{a}}$ forms a 1–dimensional $G_{\mathbf{a}}$–**Brownian motion** for which the $G_{\mathbf{a}}$–expectation coincides with $\mathbb{E}[\cdot]$.*

Example 23 *For each $0 \leq s - t$, we have*

$$\mathbb{E}[\psi(B_t - B_s)|\mathcal{H}_s] = \mathbb{E}[\psi(B_t - B_s)]$$

If ϕ is a real convex function on $\mathbb{R}$ and at least not growing too fast, then

$$\begin{aligned}
&\mathbb{E}[X\phi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\mathcal{H}_t] \\
&= X^+\mathbb{E}[\phi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\mathcal{H}_t] + X^-\mathbb{E}[-\phi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\mathcal{H}_t] \\
&= \frac{X^+}{\sqrt{2\pi(T-t)\sigma_{\mathbf{aa}^T}}} \int_{-\infty}^{\infty} \phi(x) \exp(-\frac{x^2}{2(T-t)\sigma_{\mathbf{aa}^T}})dx \\
&- \frac{X^-}{\sqrt{2\pi(T-t)|\sigma_{-\mathbf{aa}^T}|}} \int_{-\infty}^{\infty} \phi(x) \exp(-\frac{x^2}{2(T-t)|\sigma_{-\mathbf{aa}^T}|})dx.
\end{aligned}$$

In particular, for $n = 1, 2, \cdots$,

$$\begin{aligned}\mathbb{E}[|B_t^{\mathbf{a}} - B_s^{\mathbf{a}}|^n|\mathcal{H}_s] &= \mathbb{E}[|B_{t-s}^{\mathbf{a}}|^n] \\ &= \frac{1}{\sqrt{2\pi(t-s)\sigma_{\mathbf{a}\mathbf{a}^T}}}\int_{-\infty}^{\infty}|x|^n\exp(-\frac{x^2}{2(t-s)\sigma_{\mathbf{a}\mathbf{a}^T}})dx.\end{aligned}$$

But we have $\mathbb{E}[-|B_t^{\mathbf{a}} - B_s^{\mathbf{a}}|^n|\mathcal{H}_s] = \mathbb{E}[-|B_{t-s}^{\mathbf{a}}|^n]$ which is 0 when $\sigma_{-\mathbf{a}\mathbf{a}^T} = 0$ and

$$\frac{-1}{\sqrt{2\pi(t-s)|\sigma_{-\mathbf{a}\mathbf{a}^T}|}}\int_{-\infty}^{\infty}|x|^n\exp(-\frac{x^2}{2(t-s)|\sigma_{-\mathbf{a}\mathbf{a}^T}|})dx, \quad \text{if } \sigma_{-\mathbf{a}\mathbf{a}^T} < 0.$$

Exactly as in classical cases, we have $\mathbb{E}[B_t^{\mathbf{a}} - B_s^{\mathbf{a}}|\mathcal{H}_s] = 0$ and

$$\mathbb{E}[(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})^2|\mathcal{H}_s] = \sigma_{\mathbf{a}\mathbf{a}^T}(t-s), \quad \mathbb{E}[(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})^4|\mathcal{H}_s] = 3\sigma_{\mathbf{a}\mathbf{a}^T}^2(t-s)^2,$$

$$\mathbb{E}[(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})^6|\mathcal{H}_s] = 15\sigma_{\mathbf{a}\mathbf{a}^T}^3(t-s)^3, \quad \mathbb{E}[(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})^8|\mathcal{H}_s] = 105\sigma_{\mathbf{a}\mathbf{a}^T}^4(t-s)^4,$$

$$\mathbb{E}[|B_t^{\mathbf{a}} - B_s^{\mathbf{a}}||\mathcal{H}_s] = \frac{\sqrt{2(t-s)\sigma_{\mathbf{a}\mathbf{a}^T}}}{\sqrt{\pi}}, \quad \mathbb{E}[|B_t^{\mathbf{a}} - B_s^{\mathbf{a}}|^3|\mathcal{H}_s] = \frac{2\sqrt{2}[(t-s)\sigma_{\mathbf{a}\mathbf{a}^T}]^{3/2}}{\sqrt{\pi}},$$

$$\mathbb{E}[|B_t^{\mathbf{a}} - B_s^{\mathbf{a}}|^5|\mathcal{H}_s] = 8\frac{\sqrt{2}[(t-s)\sigma_{\mathbf{a}\mathbf{a}^T}]^{5/2}}{\sqrt{\pi}}.$$

Example 24 *For each $n = 1, 2, \cdots$, $0 \le t \le T$ and $X \in L_G^1(\mathcal{H}_t)$, we have*

$$\mathbb{E}[X(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\mathcal{H}_t] = X^+\mathbb{E}[(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\mathcal{H}_t] + X^-\mathbb{E}[-(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\mathcal{H}_t] = 0.$$

This with Proposition 20 yields

$$\mathbb{E}[Y + X(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\mathcal{H}_t] = \mathbb{E}[Y|\mathcal{H}_t], \quad Y \in L_G^1(\mathcal{H}).$$

We also have,

$$\begin{aligned}\mathbb{E}[X(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2|\mathcal{H}_t] &= X^+\mathbb{E}[(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2|\mathcal{H}_t] + X^-\mathbb{E}[-(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2|\mathcal{H}_t] \\ &= [X^+\sigma_{\mathbf{a}\mathbf{a}^T} + X^-\sigma_{-\mathbf{a}\mathbf{a}^T}](T-t).\end{aligned}$$

Remark 25 *It is clear that we can define an expectation $E[\cdot]$ on $L_{ip}^0(\mathcal{H})$ in the same way as in Definition 13 with the standard normal distribution $P_1^0(\cdot)$ in the place of $P_1^G(\cdot)$. If $I_d \in \Gamma$, then it follows from (3) that $P_1^0(\cdot)$ is dominated by $P_1^G(\cdot)$ in the sense*

$$P_1^0(\phi) - P_1^0(\psi) \le P_1^G(\phi - \psi).$$

Then $E[\cdot]$ can be continuously extended to $L_G^1(\mathcal{H})$. $E[\cdot]$ is a linear expectation under which $(B_t)_{t\ge0}$ behaves as a Brownian motion. We have

$$-\mathbb{E}[-X] \le E^0[X] \le \mathbb{E}[X], \quad -\mathbb{E}[-X|\mathcal{H}_t] \le E^0[X|\mathcal{H}_t] \le \mathbb{E}[X|\mathcal{H}_t]. \tag{13}$$

More generally, if $\Gamma' \subset \Gamma$, since the corresponding $P' = P^{G_{\Gamma'}}$ is dominated by $P^G = P^{G_\Gamma}$, thus the corresponding expectation $\mathbb{E}'$ is well–defined in $L_G^1(\mathcal{H})$ and $\mathbb{E}'$ is dominated by $\mathbb{E}$:

$$\mathbb{E}'[X] - \mathbb{E}'[Y] \le \mathbb{E}[X - Y], \quad X, Y \in L_G^1(\mathcal{H}).$$

Such kind of extension through the above type of domination relations was discussed in details in [44]. With this domination we then can introduce a large kind of time consistent linear or nonlinear expectations and the corresponding conditional expectations, not necessarily to be positive homogeneous and/or subadditive, as continuous functionals in $L_G^1(\mathcal{H})$. *See Example 40 for a further discussion.*

Example 26 *Since*

$$\mathbb{E}[2B_s^{\mathbf{a}}(B_t^{\mathbf{a}}-B_s^{\mathbf{a}})|\mathcal{H}_s]=\mathbb{E}[-2B_s^{\mathbf{a}}(B_t^{\mathbf{a}}-B_s^{\mathbf{a}})|\mathcal{H}_s]=0,$$

we have,

$$\begin{aligned}\mathbb{E}[(B_t^{\mathbf{a}})^2-(B_s^{\mathbf{a}})^2|\mathcal{H}_s]&=\mathbb{E}[(B_t^{\mathbf{a}}-B_s^{\mathbf{a}}+B_s^{\mathbf{a}})^2-(B_s^{\mathbf{a}})^2|\mathcal{H}_s]\\&=\mathbb{E}[(B_t^{\mathbf{a}}-B_s^{\mathbf{a}})^2+2(B_t^{\mathbf{a}}-B_s^{\mathbf{a}})B_s^{\mathbf{a}}|\mathcal{H}_s]\\&=\sigma_{\mathbf{a}\mathbf{a}^T}(t-s)\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[((B_t^{\mathbf{a}})^2-(B_s^{\mathbf{a}})^2)^2|\mathcal{H}_s]&=\mathbb{E}[\{(B_t^{\mathbf{a}}-B_s^{\mathbf{a}}+B_s^{\mathbf{a}})^2-(B_s^{\mathbf{a}})^2\}^2|\mathcal{H}_s]\\&=\mathbb{E}[\{(B_t^{\mathbf{a}}-B_s^{\mathbf{a}})^2+2(B_t^{\mathbf{a}}-B_s^{\mathbf{a}})B_s^{\mathbf{a}}\}^2|\mathcal{H}_s]\\&=\mathbb{E}[(B_t^{\mathbf{a}}-B_s^{\mathbf{a}})^4+4(B_t^{\mathbf{a}}-B_s^{\mathbf{a}})^3B_s^{\mathbf{a}}+4(B_t^{\mathbf{a}}-B_s^{\mathbf{a}})^2(B_s^{\mathbf{a}})^2|\mathcal{H}_s]\\&\le\mathbb{E}[(B_t^{\mathbf{a}}-B_s^{\mathbf{a}})^4]+4\mathbb{E}[|B_t^{\mathbf{a}}-B_s^{\mathbf{a}}|^3]|B_s^{\mathbf{a}}|+4\sigma_{\mathbf{a}\mathbf{a}^T}(t-s)(B_s^{\mathbf{a}})^2\\&=3\sigma_{\mathbf{a}\mathbf{a}^T}^2(t-s)^2+8\sqrt{\frac{2}{\pi}}[\sigma_{\mathbf{a}\mathbf{a}^T}(t-s)]^{3/2}|B_s^{\mathbf{a}}|+4\sigma_{\mathbf{a}\mathbf{a}^T}(t-s)(B_s^{\mathbf{a}})^2\end{aligned}$$

5 Itô's integral of G–Brownian motion

5.1 Bochner's integral

Definition 27 *For* $T\in\mathbb{R}_+$, *a partition* π_T *of* $[0,T]$ *is a finite ordered subset* $\pi=\{t_1,\cdots,t_N\}$ *such that* $0=t_0<t_1<\cdots<t_N=T$.

$$\mu(\pi_T)=\max\{|t_{i+1}-t_i|,i=0,1,\cdots,N-1\}.$$

We use $\pi_T^N=\{t_0^N<t_1^N<\cdots<t_N^N\}$ *to denote a sequence of partitions of* $[0,T]$ *such that* $\lim_{N\to\infty}\mu(\pi_T^N)=0$.

Let $p\geq1$ be fixed. We consider the following type of simple processes: for a given partition $\{t_0,\cdots,t_N\}=\pi_T$ of $[0,T]$, we set

$$\eta_t(\omega)=\sum_{k=0}^{N-1}\xi_k(\omega)\mathbf{I}_{[t_k,t_{k+1})}(t)$$

where $\xi_k\in L_G^p(\mathcal{H}_{t_i})$, $k=0,1,2,\cdots,N-1$ are given. The collection of these type of processes is denoted by $M_G^{p,0}(0,T)$.

Definition 28 *For an $\eta \in M_G^{1,0}(0,T)$ with $\eta_t = \sum_{k=0}^{N-1} \xi_k(\omega)\mathbf{I}_{[t_k,t_{k+1})}(t)$, the related Bochner integral is*

$$\int_0^T \eta_t(\omega)dt = \sum_{k=0}^{N-1} \xi_k(\omega)(t_{k+1} - t_k).$$

Remark 29 *We set, for each $\eta \in M_G^{1,0}(0,T)$,*

$$\tilde{\mathbb{E}}_T[\eta] := \frac{1}{T}\int_0^T \mathbb{E}[\eta_t]dt = \frac{1}{T}\sum_{k=0}^{N-1} \mathbb{E}\xi_k(\omega)(t_{k+1} - t_k).$$

It is easy to check that $\tilde{\mathbb{E}}_T : M_G^{1,0}(0,T) \longmapsto \mathbb{R}$ forms a nonlinear expectation satisfying (a)–(e) of Definition 1. We then can introduce a nature norm

$$\|\eta\|_T^1 = \tilde{\mathbb{E}}_T[|\eta|] = \frac{1}{T}\int_0^T \mathbb{E}[|\eta_t|]dt.$$

Under this norm $M_G^{1,0}(0,T)$ can extended to $M_G^1(0,T)$ which is a Banach space.

Definition 30 *For each $p \geq 1$, we denote by $M_G^p(0,T)$ the completion of $M_G^{p,0}(0,T)$ under the norm*

$$(\frac{1}{T}\int_0^T \||\eta_t|^p\| \, dt)^{1/p} = \left(\frac{1}{T}\sum_{k=0}^{N-1} \mathbb{E}[|\xi_k(\omega)|^p](t_{k+1} - t_k)\right)^{1/p}.$$

We observe that,

$$\mathbb{E}[|\int_0^T \eta_t(\omega)dt|] \leq \sum_{k=0}^{N-1} \|\xi_k(\omega)\| \, (t_{k+1} - t_k) = \int_0^T \mathbb{E}[|\eta_t|]dt. \tag{14}$$

We then have

Proposition 31 *The linear mapping $\int_0^T \eta_t(\omega)dt : M_G^{1,0}(0,T) \mapsto L_G^1(\mathcal{H}_T)$ is continuous and thus can be continuously extended to $M_G^1(0,T) \mapsto L_G^1(\mathcal{H}_T)$. We still denote this extended mapping by $\int_0^T \eta_t(\omega)dt$, $\eta \in M_G^1(0,T)$.*

Since $M_G^p(0,T) \subset M_G^1(0,T)$, for $p \geq 1$. Thus this definition holds for $\eta \in M_G^p(0,T)$.

5.2 Itô's integral of G–Brownian motion

We still use $B_t^{\mathbf{a}} := (\mathbf{a}, B_t)$ as in (12).

Definition 32 *For each $\eta \in M_G^{2,0}(0,T)$ with the form $\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega)\mathbf{I}_{[t_k,t_{k+1})}(t)$, we define*

$$I(\eta) = \int_0^T \eta(s)dB_s^{\mathbf{a}} := \sum_{k=0}^{N-1} \xi_k(B_{t_{k+1}}^{\mathbf{a}} - B_{t_k}^{\mathbf{a}}).$$

Lemma 33 *The linear mapping $I : M_G^{2,0}(0,T) \longmapsto L_G^2(\mathcal{H}_T)$ is continuous and thus can be continuously extended to $I : M_G^2(0,T) \longmapsto L_G^2(\mathcal{H}_T)$. In fact we have, for each $\eta \in M_G^{2,0}(0,T)$,*

$$\mathbb{E}[\int_0^T \eta(s)dB_s^{\mathbf{a}}] = 0, \tag{15}$$

$$\mathbb{E}[(\int_0^T \eta(s)dB_s^{\mathbf{a}})^2] \leq \sigma_{\mathbf{a}\mathbf{a}^T} \int_0^T \mathbb{E}[\eta^2(s)]ds. \tag{16}$$

Definition 34 *We define, for a fixed $\eta \in M_G^2(0,T)$ the stochastic calculus*

$$\int_0^T \eta(s)dB_s^{\mathbf{a}} := I(\eta).$$

It is clear that (15), (16) still hold for $\eta \in M_G^2(0,T)$.

Proof of Lemma 33. From Example 24, for each k,

$$\mathbb{E}[\xi_k(B_{t_{k+1}}^{\mathbf{a}} - B_{t_k}^{\mathbf{a}})|\mathcal{H}_{t_k}] = 0.$$

We have

$$\begin{aligned}\mathbb{E}[\int_0^T \eta(s)dB_s^{\mathbf{a}}] &= \mathbb{E}[\int_0^{t_{N-1}} \eta(s)dB_s^{\mathbf{a}} + \xi_{N-1}(B_{t_N}^{\mathbf{a}} - B_{t_{N-1}}^{\mathbf{a}})] \\ &= \mathbb{E}[\int_0^{t_{N-1}} \eta(s)dB_s^{\mathbf{a}} + \mathbb{E}[\xi_{N-1}(B_{t_N}^{\mathbf{a}} - B_{t_{N-1}}^{\mathbf{a}})|\mathcal{H}_{t_{N-1}}]] \\ &= \mathbb{E}[\int_0^{t_{N-1}} \eta(s)dB_s^{\mathbf{a}}].\end{aligned}$$

We then can repeat this procedure to obtain (15). We now prove (16)

$$\begin{aligned}&\mathbb{E}[\left(\int_0^T \eta(s)dB_s^{\mathbf{a}}\right)^2] = \mathbb{E}[\left(\int_0^{t_{N-1}} \eta(s)dB_s^{\mathbf{a}} + \xi_{N-1}(B_{t_N}^{\mathbf{a}} - B_{t_{N-1}}^{\mathbf{a}})\right)^2] \\ &= \mathbb{E}[\left(\int_0^{t_{N-1}} \eta(s)dB_s^{\mathbf{a}}\right)^2 \\ &+ \mathbb{E}[2\left(\int_0^{t_{N-1}} \eta(s)dB_s^{\mathbf{a}}\right)\xi_{N-1}(B_{t_N}^{\mathbf{a}} - B_{t_{N-1}}^{\mathbf{a}}) + \xi_{N-1}^2(B_{t_N}^{\mathbf{a}} - B_{t_{N-1}}^{\mathbf{a}})^2|\mathcal{H}_{t_{N-1}}]] \\ &= \mathbb{E}[\left(\int_0^{t_{N-1}} \eta(s)dB_s^{\mathbf{a}}\right)^2 + \xi_{N-1}^2\sigma_{\mathbf{a}\mathbf{a}^T}(t_N - t_{N-1})].\end{aligned}$$

Thus $\mathbb{E}[(\int_0^{t_N} \eta(s)dB_s^{\mathbf{a}})^2] \leq \mathbb{E}[\left(\int_0^{t_{N-1}} \eta(s)dB_s^{\mathbf{a}}\right)^2] + \mathbb{E}[\xi_{N-1}^2]\sigma_{\mathbf{a}\mathbf{a}^T}(t_N - t_{N-1})$. We then repeat this procedure to deduce

$$\mathbb{E}[(\int_0^T \eta(s)dB_s)^2] \leq \sigma_{\mathbf{a}\mathbf{a}^T} \sum_{k=0}^{N-1} \mathbb{E}[(\xi_k)^2](t_{k+1} - t_k) = \int_0^T \mathbb{E}[(\eta(t))^2]dt.$$

■

We list some main property of the Itô's integral of G–Brownian motion. We denote for some $0 \leq s \leq t \leq T$,

$$\int_s^t \eta_u dB_u^{\mathbf{a}} := \int_0^T \mathbf{I}_{[s,t]}(u)\eta_u dB_u^{\mathbf{a}}.$$

We have

Proposition 35 *Let $\eta, \theta \in M_G^2(0,T)$ and let $0 \leq s \leq r \leq t \leq T$. Then in $L_G^1(\mathcal{H}_T)$ we have*
(i) $\int_s^t \eta_u dB_u^{\mathbf{a}} = \int_s^r \eta_u dB_u^{\mathbf{a}} + \int_r^t \eta_u dB_u^{\mathbf{a}}$.
(ii) $\int_s^t (\alpha\eta_u + \theta_u) dB_u^{\mathbf{a}} = \alpha \int_s^t \eta_u dB_u^{\mathbf{a}} + \int_s^t \theta_u dB_u^{\mathbf{a}}$, if α is bounded and in $L_G^1(\mathcal{H}_s)$,
(iii) $\mathbb{E}[X + \int_r^T \eta_u dB_u^{\mathbf{a}} | \mathcal{H}_s] = \mathbb{E}[X|\mathcal{H}_s]$, $\forall X \in L_G^1(\mathcal{H})$
(iv) $\mathbb{E}[(\int_r^T \eta_u dB_u^{\mathbf{a}})^2 | \mathcal{H}_s] \leq \sigma_{\mathbf{a}\mathbf{a}^T} \int_r^T \mathbb{E}[|\eta_u|^2 | \mathcal{H}_s] du$

5.3 Quadratic variation process of G–Brownian motion

We now consider the quadratic variation of G–Brownian motion. It concentrically reflects the characteristic of the 'uncertainty' part of the G-Brownian motion B. This makes a major difference from the classical Brownian motion.

Let π_t^N, $N = 1, 2, \cdots$, be a sequence of partitions of $[0,t]$. We consider

$$\begin{aligned}(B_t^{\mathbf{a}})^2 &= \sum_{k=0}^{N-1} [(B_{t_{k+1}^N}^{\mathbf{a}})^2 - (B_{t_k^N}^{\mathbf{a}})^2] \\ &= \sum_{k=0}^{N-1} 2B_{t_k^N}^{\mathbf{a}} (B_{t_{k+1}^N}^{\mathbf{a}} - B_{t_k^N}^{\mathbf{a}}) + \sum_{k=0}^{N-1} (B_{t_{k+1}^N}^{\mathbf{a}} - B_{t_k^N}^{\mathbf{a}})^2\end{aligned}$$

As $\mu(\pi_t^N) = \max_{0 \leq k \leq N-1}(t_{k+1}^N - t_k^N) \to 0$, the first term of the right side tends to $\int_0^t B_s^{\mathbf{a}} dB_s^{\mathbf{a}}$. The second term must converge. We denote its limit by $\langle B^{\mathbf{a}} \rangle_t$, i.e.,

$$\langle B^{\mathbf{a}} \rangle_t = \lim_{\mu(\pi_t^N) \to 0} \sum_{k=0}^{N-1} (B_{t_{k+1}^N}^{\mathbf{a}} - B_{t_k^N}^{\mathbf{a}})^2 = (B_t^{\mathbf{a}})^2 - 2\int_0^t B_s^{\mathbf{a}} dB_s^{\mathbf{a}}. \qquad (17)$$

By the above construction, $\langle B^{\mathbf{a}} \rangle_t$, $t \geq 0$, is an increasing process with $\langle B^{\mathbf{a}} \rangle_0 = 0$. We call it the **quadratic variation process** of the G–Brownian motion $B^{\mathbf{a}}$. Clearly $\langle B^{\mathbf{a}} \rangle$ is an increasing process. It is also clear that, for each $0 \leq s \leq t$ and for each smooth real function ψ such that $\psi(\langle B^{\mathbf{a}} \rangle_{t-s}) \in L_G^1(\mathcal{H}_{t-s})$ we have $\mathbb{E}[\psi(\langle B^{\mathbf{a}} \rangle_{t-s})] = \mathbb{E}[\psi(\langle B^{\mathbf{a}} \rangle_t - \langle B^{\mathbf{a}} \rangle_s)]$. We also have

$$\langle B^{\mathbf{a}} \rangle_t = \langle B^{-\mathbf{a}} \rangle_t = \langle -B^{\mathbf{a}} \rangle_t .$$

It is important to keep in mind that $\langle B^{\mathbf{a}} \rangle_t$ is not a deterministic process unless the case $\sigma_{\mathbf{a}\mathbf{a}^T} = -\sigma_{-\mathbf{a}\mathbf{a}^T}$ and thus $B^{\mathbf{a}}$ becomes a classical Brownian motion. In fact we have

Lemma 36 *For each* $0 \le s \le t < \infty$

$$\mathbb{E}[\langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s |\mathcal{H}_s] = \sigma_{\mathbf{a}\mathbf{a}^T}(t-s), \tag{18}$$
$$\mathbb{E}[-(\langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s)|\mathcal{H}_s] = \sigma_{-\mathbf{a}\mathbf{a}^T}(t-s). \tag{19}$$

Proof. By the definition of $\langle B^{\mathbf{a}}\rangle$ and Proposition 35-(iii), then Example 26,

$$\begin{aligned}\mathbb{E}[\langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s |\mathcal{H}_s] &= \mathbb{E}[(B^{\mathbf{a}}_t)^2 - (B^{\mathbf{a}}_s)^2 - 2\int_s^t B^{\mathbf{a}}_u dB^{\mathbf{a}}_u |\mathcal{H}_s] \\ &= \mathbb{E}[(B^{\mathbf{a}}_t)^2 - (B^{\mathbf{a}}_s)^2 |\mathcal{H}_s] = \sigma_{\mathbf{a}\mathbf{a}^T}(t-s).\end{aligned}$$

We then have (18). (19) can be proved analogously by using the equality $\mathbb{E}[-((B^{\mathbf{a}}_t)^2 - (B^{\mathbf{a}}_s)^2)|\mathcal{H}_s] = \sigma_{-\mathbf{a}\mathbf{a}^T}(t-s)$. ■

An interesting new phenomenon of our G-Brownian motion is that its quadratic process $\langle B\rangle$ also has independent increments. In fact, we have

Lemma 37 *An increment of* $\langle B^{\mathbf{a}}\rangle$ *is the quadratic variation of the corresponding increment of* $B^{\mathbf{a}}$*, i.e., for each fixed* $s \ge 0$,

$$\langle B^{\mathbf{a}}\rangle_{t+s} - \langle B^{\mathbf{a}}\rangle_s = \langle (B^s)^{\mathbf{a}}\rangle_t$$

where $B^s_t = B_{t+s} - B_s$, $t \ge 0$ *and* $(B^s)^{\mathbf{a}}_t = (\mathbf{a}, B^t_s)$.

Proof.

$$\begin{aligned}\langle B^{\mathbf{a}}\rangle_{t+s} - \langle B^{\mathbf{a}}\rangle_s &= (B^{\mathbf{a}}_{t+s})^2 - 2\int_0^{t+s} B^{\mathbf{a}}_u dB^{\mathbf{a}}_u - \left((B^{\mathbf{a}}_s)^2 - 2\int_0^s B^{\mathbf{a}}_u dB^{\mathbf{a}}_u\right) \\ &= (B^{\mathbf{a}}_{t+s} - B^{\mathbf{a}}_s)^2 - 2\int_s^{t+s}(B^{\mathbf{a}}_u - B^{\mathbf{a}}_s)dB^{\mathbf{a}}_u \\ &= (B^{\mathbf{a}}_{t+s} - B^{\mathbf{a}}_s)^2 - 2\int_s^t (B^{\mathbf{a}}_{s+u} - B^{\mathbf{a}}_s)d(B^{\mathbf{a}}_u - B^{\mathbf{a}}_s) \\ &= \langle (B^s)^{\mathbf{a}}\rangle_t .\end{aligned}$$

■

Lemma 38 *We have*

$$\mathbb{E}[\langle B^{\mathbf{a}}\rangle_t^2] = \mathbb{E}[(\langle B^{\mathbf{a}}\rangle_{t+s} - \langle B^{\mathbf{a}}\rangle_s)^2 |\mathcal{H}_s] = \sigma^2_{\mathbf{a}\mathbf{a}^T} t^2, \quad s, t \ge 0. \tag{20}$$

Proof. We set $\phi(t) := \mathbb{E}[\langle B^{\mathbf{a}}\rangle_t^2]$.

$$\begin{aligned}\phi(t) &= \mathbb{E}[\{(B^{\mathbf{a}}_t)^2 - 2\int_0^t B^{\mathbf{a}}_u dB^{\mathbf{a}}_u\}^2] \\ &\le 2\mathbb{E}[(B^{\mathbf{a}}_t)^4] + 8\mathbb{E}[(\int_0^t B^{\mathbf{a}}_u dB^{\mathbf{a}}_u)^2] \\ &\le 6\sigma^2_{\mathbf{a}\mathbf{a}^T} t^2 + 8\sigma_{\mathbf{a}\mathbf{a}^T}\int_0^t \mathbb{E}[(B^{\mathbf{a}}_u)^2]du \\ &= 10\sigma^2_{\mathbf{a}\mathbf{a}^T} t^2.\end{aligned}$$

This also implies $\mathbb{E}[(\langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s)^2] = \phi(t-s) \leq 10\sigma^2_{\mathbf{aa}^T}(t-s)^2$. For each $s \in [0,t)$,

$$\begin{aligned}\phi(t) &= \mathbb{E}[(\langle B^{\mathbf{a}}\rangle_s + \langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s)^2] \\ &\leq \mathbb{E}[(\langle B^{\mathbf{a}}\rangle_s)^2] + \mathbb{E}[(\langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s)^2] + 2\mathbb{E}[(\langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s)\langle B^{\mathbf{a}}\rangle_s] \\ &= \phi(s) + \phi(t-s) + 2\mathbb{E}[\mathbb{E}[(\langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s)|\mathcal{H}_s]\langle B^{\mathbf{a}}\rangle_s] \\ &= \phi(s) + \phi(t-s) + 2\sigma^2_{\mathbf{aa}^T} s(t-s).\end{aligned}$$

We set $\delta_N = t/N$, $t_k^N = kt/N = k\delta_N$ for a positive integer N. By the above inequalities

$$\begin{aligned}\phi(t_N^N) &\leq \phi(t_{N-1}^N) + \phi(\delta_N) + 2\sigma^2_{\mathbf{aa}^T} t_{N-1}^N \delta_N \\ &\leq \phi(t_{N-2}^N) + 2\phi(\delta_N) + 2\sigma^2_{\mathbf{aa}^T}(t_{N-1}^N + t_{N-2}^N)\delta_N \\ &\vdots\end{aligned}$$

We then have

$$\phi(t) \leq N\phi(\delta_N) + 2\sigma^2_{\mathbf{aa}^T}\sum_{k=0}^{N-1} t_k^N \delta_N \leq 14t^2\sigma^2_{\mathbf{aa}^T}/N + 2\sigma^2_{\mathbf{aa}^T}\sum_{k=0}^{N-1} t_k^N \delta_N.$$

Let $N \to \infty$ we have $\phi(t) \leq 2\sigma^2_{\mathbf{aa}^T}\int_0^t sds = \sigma^2_{\mathbf{aa}^T}t^2$. Thus $\mathbb{E}[\langle B^{\mathbf{a}}\rangle_t^2] \leq \sigma^2_{\mathbf{aa}^T}t^2$. This with $\mathbb{E}[\langle B^{\mathbf{a}}\rangle_t^2] \geq E^0[\langle B^{\mathbf{a}}\rangle_t^2] = \sigma^2_{\mathbf{aa}^T}t^2$ implies (20). In the last step, the classical normal distribution P_1^0, or $N(0, \gamma_0\gamma_0^T)$, $\gamma_0 \in \Gamma$, is chosen such that

$$tr[\gamma_0\gamma_0^T\mathbf{aa}^T] = \sigma^2_{\mathbf{aa}^T} = \sup_{\gamma\in\Gamma} tr[\gamma\gamma^T\mathbf{aa}^T].$$

■

Similarly we have

$$\begin{aligned}\mathbb{E}[(\langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s)^3|\mathcal{H}_s] &= \sigma^3_{\mathbf{aa}^T}(t-s)^3, \\ \mathbb{E}[(\langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s)^4|\mathcal{H}_s] &= \sigma^4_{\mathbf{aa}^T}(t-s)^4.\end{aligned} \tag{21}$$

Proposition 39 *Let $0 \leq s \leq t$, $\xi \in L_G^1(\mathcal{H}_s)$, $X \in L_G^1(\mathcal{H})$. Then*

$$\begin{aligned}\mathbb{E}[X + \xi((B_t^{\mathbf{a}})^2 - (B_s^{\mathbf{a}})^2)] &= \mathbb{E}[X + \xi(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})^2] \\ &= \mathbb{E}[X + \xi(\langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s)].\end{aligned}$$

Proof. By (17) and applying Proposition 20, we have

$$\begin{aligned}\mathbb{E}[X + \xi((B_t^{\mathbf{a}})^2 - (B_s^{\mathbf{a}})^2)] &= \mathbb{E}[X + \xi(\langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s + 2\int_s^t B_u^{\mathbf{a}} dB_u^{\mathbf{a}})] \\ &= \mathbb{E}[X + \xi(\langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s)].\end{aligned}$$

We also have

$$\begin{aligned}\mathbb{E}[X+\xi((B_t^{\mathbf{a}})^2-(B_s^{\mathbf{a}})^2)] &= \mathbb{E}[X+\xi\{(B_t^{\mathbf{a}}-B_s^{\mathbf{a}})^2+2(B_t^{\mathbf{a}}-B_s^{\mathbf{a}})B_s^{\mathbf{a}}\}] \\ &= \mathbb{E}[X+\xi(B_t^{\mathbf{a}}-B_s^{\mathbf{a}})^2].\end{aligned}$$

■

Example 40 *We assume that in a financial market a stock price* $(S_t)_{t\geq 0}$ *is observed. Let* $B_t=\log(S_t)$, $t\geq 0$, *be a 1-dimensional G-Brownian motion* $(d=1)$ *with* $\Gamma=[\sigma_*,\sigma^*]$, *with fixed* $\sigma_*\in[0,\frac{1}{2})$ *and* $\sigma^*\in[1,\infty)$. *Two traders* a *and* b *in a same bank are using their own statistics to price a contingent claim* $X=\langle B\rangle_T$ *with maturity* T. *Suppose, for example, under the probability measure* $\mathbb{P}_a$ *of* a, B *is a (classical) Brownian motion whereas under* $\mathbb{P}_b$ *of* b, $\frac{1}{2}B$ *is a Brownian motion, here* $\mathbb{P}_a$ *(resp.* $\mathbb{P}_b$*) is a classical probability measure with its linear expectation* $\mathbb{E}^a$ *(resp.* $\mathbb{E}^b$ *) generated by the heat equation* $\partial_t u=\frac{1}{2}\partial_{xx}^2 u$ *(resp.* $\partial_t u=\frac{1}{4}\partial_{xx}^2 u$*). Since* $\mathbb{E}^a$ *and* $\mathbb{E}^b$ *are both dominated by* $\mathbb{E}$ *in the sense of (3), they can be both well–defined as a linear bounded functional in* $L_G^1(\mathcal{H})$. *This framework cannot be provided by just using a classical probability space because it is known that* $\langle B\rangle_T=T$, $\mathbb{P}^a$*–a.s., and* $\langle B\rangle_T=\frac{T}{4}$, $\mathbb{P}^b$*–a.s. Thus there is no probability measure on* Ω *with respect to which* P_a *and* P_b *are both absolutely continuous. Practically this sublinear expectation* $\mathbb{E}$ *provides a realistic tool of dynamic risk measure for a risk supervisor of the traders a and b: given a risk position* $X\in L_G^1(\mathcal{H}_T)$ *we always have* $\mathbb{E}[-X|\mathcal{H}_t]\geq\mathbb{E}^a[-X|\mathcal{H}_t]\vee\mathbb{E}^b[-X|\mathcal{H}_t]$ *for the loss* $-X$ *of this position. The meaning is that the supervisor uses a more sensitive risk measure. Clearly no linear expectation can play this role. The subset* Γ *represents the uncertainty of the volatility model of a risk regulator. The lager the subset* Γ, *the bigger the uncertainty, thus the stronger the corresponding* $\mathbb{E}$.

It is worth to consider to create a hierarchic and dynamic risk control system for a bank, or a banking system, in which the Chief Risk Officer (CRO) uses $\mathbb{E}=\mathbb{E}^G$ *for his risk measure and the Risk Officer the* i*th division of the bank uses* $\mathbb{E}^i=\mathbb{E}^{G_i}$ *for his one, where*

$$G(A)=\frac{1}{2}\sup_{\gamma\in\Gamma}tr[\gamma\gamma^T A],\quad G_i(A)=\frac{1}{2}\sup_{\gamma\in\Gamma_i}tr[\gamma\gamma^T A],\ \Gamma_i\subset\Gamma,\ \ i=1,\cdots,I.$$

Thus $\mathbb{E}^i$ *is dominated by* $\mathbb{E}$ *for each* i. *For a large banking system we can even consider to create* $\mathbb{E}^{ij}=\mathbb{E}^{G_{ij}}$ *for its* (i,j)*th sub-division. The reasoning is: in general, a risk regulator's statistics and knowledge of a specific risk position* X *are less than a trader who is concretely involved in the business of the product* X.

To define the integration of a process $\eta\in M_G^1(0,T)$ with respect to $d\langle B^{\mathbf{a}}\rangle$, we first define a mapping:

$$Q_{0,T}(\eta)=\int_0^T\eta(s)d\langle B^{\mathbf{a}}\rangle_s:=\sum_{k=0}^{N-1}\xi_k(\langle B^{\mathbf{a}}\rangle_{t_{k+1}}-\langle B^{\mathbf{a}}\rangle_{t_k}):M_G^{1,0}(0,T)\mapsto L^1(\mathcal{H}_T).$$

Lemma 41 *For each $\eta \in M_G^{1,0}(0,T)$,*

$$\mathbb{E}[|Q_{0,T}(\eta)|] \leq \sigma_{\mathbf{aa}^T} \int_0^T \mathbb{E}[|\eta_s|]ds, \tag{22}$$

Thus $Q_{0,T} : M_G^{1,0}(0,T) \mapsto L^1(\mathcal{H}_T)$ is a continuous linear mapping. Consequently, $Q_{0,T}$ can be uniquely extended to $M_G^1(0,T)$. We still denote this mapping by

$$\int_0^T \eta(s) d\left\langle B^{\mathbf{a}}\right\rangle_s = Q_{0,T}(\eta), \quad \eta \in M_G^1(0,T).$$

We still have

$$\mathbb{E}[|\int_0^T \eta(s) d\left\langle B^{\mathbf{a}}\right\rangle_s|] \leq \sigma_{\mathbf{aa}^T} \int_0^T \mathbb{E}[|\eta_s|]ds, \quad \forall \eta \in M_G^1(0,T). \tag{23}$$

Proof. By applying Lemma 36, (22) can be checked as follows:

$$\begin{aligned}
\mathbb{E}[|\sum_{k=0}^{N-1} \xi_k(\left\langle B^{\mathbf{a}}\right\rangle_{t_{k+1}} - \left\langle B^{\mathbf{a}}\right\rangle_{t_k})|] &\leq \sum_{k=0}^{N-1} \mathbb{E}[|\xi_k| \cdot \mathbb{E}[\left\langle B^{\mathbf{a}}\right\rangle_{t_{k+1}} - \left\langle B^{\mathbf{a}}\right\rangle_{t_k} |\mathcal{H}_{t_k}]] \\
&= \sum_{k=0}^{N-1} \mathbb{E}[|\xi_k|]\sigma_{\mathbf{aa}^T}(t_{k+1} - t_k) \\
&= \sigma_{\mathbf{aa}^T} \int_0^T \mathbb{E}[|\eta_s|]ds.
\end{aligned}$$

■

We have the following isometry

Proposition 42 *Let $\eta \in M_G^2(0,T)$*

$$\mathbb{E}[(\int_0^T \eta(s) dB_s^{\mathbf{a}})^2] = \mathbb{E}[\int_0^T \eta^2(s) d\left\langle B^{\mathbf{a}}\right\rangle_s] \tag{24}$$

Proof. We first consider $\eta \in M_G^{2,0}(0,T)$ with the form

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega)\mathbf{I}_{[t_k,t_{k+1})}(t)$$

and thus $\int_0^T \eta(s)dB_s^{\mathbf{a}} := \sum_{k=0}^{N-1} \xi_k(B_{t_{k+1}}^{\mathbf{a}} - B_{t_k}^{\mathbf{a}})$. By Proposition 20 we have

$$\mathbb{E}[X + 2\xi_k(B_{t_{k+1}}^{\mathbf{a}} - B_{t_k}^{\mathbf{a}})\xi_l(B_{t_{l+1}}^{\mathbf{a}} - B_{t_l}^{\mathbf{a}})] = \mathbb{E}[X], \text{ for } X \in L_G^1(\mathcal{H}),\ l \neq k.$$

Thus

$$\mathbb{E}[(\int_0^T \eta(s) dB_s^{\mathbf{a}})^2] = \mathbb{E}[\left(\sum_{k=0}^{N-1} \xi_k(B_{t_{k+1}}^{\mathbf{a}} - B_{t_k}^{\mathbf{a}})\right)^2] = \mathbb{E}[\sum_{k=0}^{N-1} \xi_k^2(B_{t_{k+1}}^{\mathbf{a}} - B_{t_k}^{\mathbf{a}})^2].$$

This with Proposition 39, it follows that

$$\mathbb{E}[(\int_0^T \eta(s)dB_s^{\mathbf{a}})^2] = \mathbb{E}[\sum_{k=0}^{N-1} \xi_k^2(\langle B^{\mathbf{a}}\rangle_{t_{k+1}} - \langle B^{\mathbf{a}}\rangle_{t_k})] = \mathbb{E}[\int_0^T \eta^2(s)d\langle B^{\mathbf{a}}\rangle_s].$$

Thus (24) holds for $\eta \in M_G^{2,0}(0,T)$. We thus can continuously extend this equality to the case $\eta \in M_G^2(0,T)$ and obtain (24). ■

5.4 Mutual variation processes for G–Brownian motion

Let $\mathbf{a} = (a_1, \cdots, a_d)^T$ and $\bar{\mathbf{a}} = (\bar{a}_1, \cdots, \bar{a}_d)^T$ be two given vectors in $\mathbb{R}^d$. We then have their quadratic variation process $\langle B^{\mathbf{a}}\rangle$ and $\langle B^{\bar{\mathbf{a}}}\rangle$. We then can define their mutual variation process by

$$\begin{aligned}\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t &:= \frac{1}{4}[\langle B^{\mathbf{a}} + B^{\bar{\mathbf{a}}}\rangle_t - \langle B^{\mathbf{a}} - B^{\bar{\mathbf{a}}}\rangle_t] \\ &= \frac{1}{4}[\langle B^{\mathbf{a}+\bar{\mathbf{a}}}\rangle_t - \langle B^{\mathbf{a}-\bar{\mathbf{a}}}\rangle_t].\end{aligned}$$

Since $\langle B^{\mathbf{a}-\bar{\mathbf{a}}}\rangle = \langle B^{\bar{\mathbf{a}}-\mathbf{a}}\rangle = \langle -B^{\mathbf{a}-\bar{\mathbf{a}}}\rangle$, we see that $\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t = \langle B^{\bar{\mathbf{a}}}, B^{\mathbf{a}}\rangle_t$. In particular we have $\langle B^{\mathbf{a}}, B^{\mathbf{a}}\rangle = \langle B^{\mathbf{a}}\rangle$. Let π_t^N, $N = 1, 2, \cdots$, be a sequence of partitions of $[0,t]$. We observe that

$$\sum_{k=0}^{N-1}(B_{t_{k+1}^N}^{\mathbf{a}} - B_{t_k^N}^{\mathbf{a}})(B_{t_{k+1}^N}^{\bar{\mathbf{a}}} - B_{t_k^N}^{\bar{\mathbf{a}}}) = \frac{1}{4}\sum_{k=0}^{N-1}[(B_{t_{k+1}}^{\mathbf{a}+\bar{\mathbf{a}}} - B_{t_k}^{\mathbf{a}+\bar{\mathbf{a}}})^2 - (B_{t_{k+1}}^{\mathbf{a}-\bar{\mathbf{a}}} - B_{t_k}^{\mathbf{a}-\bar{\mathbf{a}}})^2].$$

Thus as $\mu(\pi_t^N) \to 0$, we have

$$\lim_{N\to 0}\sum_{k=0}^{N-1}(B_{t_{k+1}^N}^{\mathbf{a}} - B_{t_k^N}^{\mathbf{a}})(B_{t_{k+1}^N}^{\bar{\mathbf{a}}} - B_{t_k^N}^{\bar{\mathbf{a}}}) = \langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t.$$

We also have

$$\begin{aligned}\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t &= \frac{1}{4}[\langle B^{\mathbf{a}+\bar{\mathbf{a}}}\rangle_t - \langle B^{\mathbf{a}-\bar{\mathbf{a}}}\rangle_t] \\ &= \frac{1}{4}[(B_t^{\mathbf{a}+\bar{\mathbf{a}}})^2 - 2\int_0^t B_s^{\mathbf{a}+\bar{\mathbf{a}}}dB_s^{\mathbf{a}+\bar{\mathbf{a}}} - (B_t^{\mathbf{a}-\bar{\mathbf{a}}})^2 + 2\int_0^t B_s^{\mathbf{a}-\bar{\mathbf{a}}}dB_s^{\mathbf{a}-\bar{\mathbf{a}}}] \\ &= B_t^{\mathbf{a}}B_t^{\bar{\mathbf{a}}} - \int_0^t B_s^{\mathbf{a}}dB_s^{\bar{\mathbf{a}}} - \int_0^t B_s^{\bar{\mathbf{a}}}dB_s^{\mathbf{a}}.\end{aligned}$$

Now for each $\eta \in M_G^1(0,T)$ we can consistently define

$$\int_0^T \eta_s d\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_s = \frac{1}{4}\int_0^T \eta_s d\langle B^{\mathbf{a}+\bar{\mathbf{a}}}\rangle_s - \frac{1}{4}\int_0^T \eta_s \langle B^{\mathbf{a}-\bar{\mathbf{a}}}\rangle_s.$$

Lemma 43 *Let $\eta^N \in M_G^{1,0}(0,T)$, $N = 1, 2, \cdots$, be of form*

$$\eta_t^N(\omega) = \sum_{k=0}^{N-1} \xi_k^N(\omega)\mathbf{I}_{[t_k^N, t_{k+1}^N)}(t)$$

with $\mu(\pi_T^N) \to 0$ and $\eta^N \to \eta$ in $M_G^1(0,T)$ as $N \to \infty$. Then we have the following convergence in $L_G^1(\mathcal{H}_T)$:

$$\int_0^T \eta^N(s)d\left\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\right\rangle_s := \sum_{k=0}^{N-1} \xi_k^N(B_{t_{k+1}^N}^{\mathbf{a}} - B_{t_k^N}^{\mathbf{a}})(B_{t_{k+1}^N}^{\bar{\mathbf{a}}} - B_{t_k^N}^{\bar{\mathbf{a}}})$$
$$\to \int_0^T \eta(s)d\left\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\right\rangle_s.$$

5.5 Itô's formula for G–Brownian motion

We have the corresponding Itô's formula of $\Phi(X_t)$ for a "G-Itô process" X. For simplification, we only treat the case where the function Φ is sufficiently regular. For notational simplification, we denote $B^i = B^{\mathbf{e}_i}$, the i–th coordinate of the G–Brownian motion B, under a given orthonormal base $(\mathbf{e}_1, \cdots, \mathbf{e}_d)$ of $\mathbb{R}^d$.

Lemma 44 *Let $\Phi \in C^2(\mathbb{R}^n)$ be bounded with bounded derivatives and $\{\partial^2_{x^\mu x^\nu}\Phi\}_{\mu,\nu=1}^n$ are uniformly Lipschitz. Let $s \in [0,T]$ be fixed and let $X = (X^1, \cdots, X^n)^T$ be an n–dimensional process on $[s,T]$ of the form*

$$X_t^\nu = X_s^\nu + \alpha^\nu(t-s) + \eta^{\nu ij}(\left\langle B^i, B^j\right\rangle_t - \left\langle B^i, B^j\right\rangle_s) + \beta^{\nu j}(B_t^j - B_s^j),$$

where, for $\nu = 1, \cdots, n$, $i, j = 1, \cdots, d$, α^ν, $\eta^{\nu ij}$ and $\beta^{\nu ij}$, are bounded elements of $L_G^2(\mathcal{H}_s)$ and $X_s = (X_s^1, \cdots, X_s^n)^T$ is a given $\mathbb{R}^n$–vector in $L_G^2(\mathcal{H}_s)$. Then we have

$$\Phi(X_t) - \Phi(X_s) = \int_s^t \partial_{x^\nu}\Phi(X_u)\beta^{\nu j}dB_u^j + \int_s^t \partial_{x_\nu}\Phi(X_u)\alpha^\nu du \tag{25}$$
$$+ \int_s^t [\partial_{x^\nu}\Phi(X_u)\eta^{\nu ij} + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u)\beta^{\nu i}\beta^{\nu j}]d\left\langle B^i, B^j\right\rangle_u.$$

Here we use the Einstein convention, i.e., the above repeated indices μ, ν, i and j (but not k) imply the summation.

Proof. For each positive integer N we set $\delta = (t-s)/N$ and take the partition

$$\pi_{[s,t]}^N = \{t_0^N, t_1^N, \cdots, t_N^N\} = \{s, s+\delta, \cdots, s+N\delta = t\}.$$

We have

$$
\begin{aligned}
\Phi(X_t) - \Phi(X_s) &= \sum_{k=0}^{N-1} [\Phi(X_{t_{k+1}^N}) - \Phi(X_{t_k^N})] \\
&= \sum_{k=0}^{N-1} [\partial_{x^\mu}\Phi(X_{t_k^N})(X^\mu_{t_{k+1}^N} - X^\mu_{t_k^N}) \\
&+ \frac{1}{2}[\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})(X^\mu_{t_{k+1}^N} - X^\mu_{t_k^N})(X^\nu_{t_{k+1}^N} - X^\nu_{t_k^N}) + \eta_k^N]] \qquad (26)
\end{aligned}
$$

where

$$
\eta_k^N = [\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N} + \theta_k(X_{t_{k+1}^N} - X_{t_k^N})) - \partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})](X^\mu_{t_{k+1}^N} - X^\mu_{t_k^N})(X^\nu_{t_{k+1}^N} - X^\nu_{t_k^N})
$$

with $\theta_k \in [0,1]$. We have

$$
\begin{aligned}
\mathbb{E}[|\eta_k^N|] &= \mathbb{E}[|[\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N} + \theta_k(X_{t_{k+1}^N} - X_{t_k^N})) - \partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})] \\
&\times (X^\mu_{t_{k+1}^N} - X^\mu_{t_k^N})(X^\nu_{t_{k+1}^N} - X^\nu_{t_k^N})|] \\
&\le c\mathbb{E}[|X_{t_{k+1}^N} - X_{t_k^N}|^3] \le C[\delta^3 + \delta^{3/2}]
\end{aligned}
$$

where c is the Lipschitz constant of $\{\partial^2_{x^\mu x^\nu}\Phi\}_{\mu,\nu=1}^d$. In the last step we use Example 23 and (21). Thus $\sum_k \mathbb{E}[|\eta_k^N|] \to 0$. The rest terms in the summation of the right side of (26) are $\xi_t^N + \zeta_t^N$ with

$$
\begin{aligned}
\xi_t^N &= \sum_{k=0}^{N-1} \{\partial_{x^\mu}\Phi(X_{t_k^N})[\alpha^\mu(t_{k+1}^N - t_k^N) + \eta^{\mu ij}(\langle B^i, B^j\rangle_{t_{k+1}^N} - \langle B^i, B^j\rangle_{t_k^N}) \\
&+ \beta^{\mu j}(B^j_{t_{k+1}^N} - B^j_{t_k^N})] + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})\beta^{\mu i}\beta^{\nu j}(B^i_{t_{k+1}^N} - B^i_{t_k^N})(B^j_{t_{k+1}^N} - B^j_{t_k^N})\}
\end{aligned}
$$

and

$$
\begin{aligned}
\zeta_t^N &= \frac{1}{2}\sum_{k=0}^{N-1} \partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})[\alpha^\mu(t_{k+1}^N - t_k^N) + \eta^{\mu ij}(\langle B^i, B^j\rangle_{t_{k+1}^N} - \langle B^i, B^j\rangle_{t_k^N})] \\
&\times [\alpha^\nu(t_{k+1}^N - t_k^N) + \eta^{\nu lm}(\langle B^l, B^m\rangle_{t_{k+1}^N} - \langle B^l, B^m\rangle_{t_k^N})] \\
&+ [\alpha^\mu(t_{k+1}^N - t_k^N) + \eta^{\mu ij}(\langle B^i, B^j\rangle_{t_{k+1}^N} - \langle B^i, B^j\rangle_{t_k^N})]\beta^{\nu l}(B^l_{t_{k+1}^N} - B^l_{t_k^N})
\end{aligned}
$$

We observe that, for each $u \in [t_k^N, t_{k+1}^N)$

$$
\begin{aligned}
&\mathbb{E}[|\partial_{x^\mu}\Phi(X_u) - \sum_{k=0}^{N-1}\partial_{x^\mu}\Phi(X_{t_k^N})\mathbf{I}_{[t_k^N, t_{k+1}^N)}(u)|^2] \\
&= \mathbb{E}[|\partial_{x^\mu}\Phi(X_u) - \partial_{x^\mu}\Phi(X_{t_k^N})|^2] \\
&\le c^2\mathbb{E}[|X_u - X_{t_k^N}|^2] \le C[\delta + \delta^2].
\end{aligned}
$$

Thus $\sum_{k=0}^{N-1} \partial_{x^\mu}\Phi(X_{t_k^N})\mathbf{I}_{[t_k^N,t_{k+1}^N)}(\cdot)$ tends to $\partial_{x^\mu}\Phi(X_\cdot)$ in $M_G^2(0,T)$. Similarly,

$$\sum_{k=0}^{N-1} \partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})\mathbf{I}_{[t_k^N,t_{k+1}^N)}(\cdot) \to \partial^2_{x^\mu x^\nu}\Phi(X_\cdot), \text{ in } \ M_G^2(0,T).$$

Let $N\to\infty$, by Lemma 43 as well as the definitions of the integrations of dt, dB_t and $d\langle B\rangle_t$, the limit of ξ_t^N in $L_G^2(\mathcal{H}_t)$ is just the right hand side of (25). By the next Remark, we also have $\zeta_t^N\to 0$ in $L_G^2(\mathcal{H}_t)$. We then have proved (25). ■

Remark 45 *In the proof of $\zeta_t^N\to 0$ in $L_G^2(\mathcal{H}_t)$, we use the following estimates: for $\psi^N\in M_G^{1,0}(0,T)$ such that $\psi_t^N=\sum_{k=0}^{N-1}\xi_{t_k}^N\mathbf{I}_{[t_k^N,t_{k+1}^N)}(t)$, and $\pi_T^N=\{0\le t_0,\cdots,t_N=T\}$ with $\lim_{N\to\infty}\mu(\pi_T^N)=0$ and $\sum_{k=0}^{N-1}\mathbb{E}[|\xi_{t_k}^N|](t_{k+1}^N-t_k^N)\le C$, for all $N=1,2,\cdots$, we have $\mathbb{E}[|\sum_{k=0}^{N-1}\xi_k^N(t_{k+1}^N-t_k^N)^2]\to 0$ and, for any fixed $\mathbf{a},\bar{\mathbf{a}}\in\mathbb{R}^d$,*

$$\begin{aligned}\mathbb{E}[|\sum_{k=0}^{N-1}\xi_k^N(\langle B^{\mathbf{a}}\rangle_{t_{k+1}^N}-\langle B^{\mathbf{a}}\rangle_{t_k^N})^2|] &\le \sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|\cdot\mathbb{E}[(\langle B^{\mathbf{a}}\rangle_{t_{k+1}^N}-\langle B^{\mathbf{a}}\rangle_{t_k^N})^2|\mathcal{H}_{t_k^N}]] \\ &= \sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|]\sigma^2_{\mathbf{a}\mathbf{a}^T}(t_{k+1}^N-t_k^N)^2\to 0,\end{aligned}$$

$$\begin{aligned}&\mathbb{E}[|\sum_{k=0}^{N-1}\xi_k^N(\langle B^{\mathbf{a}}\rangle_{t_{k+1}^N}-\langle B^{\mathbf{a}}\rangle_{t_k^N})(t_{k+1}^N-t_k^N)|] \\ &\le \sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|(t_{k+1}^N-t_k^N)\cdot\mathbb{E}[(\langle B^{\mathbf{a}}\rangle_{t_{k+1}^N}-\langle B^{\mathbf{a}}\rangle_{t_k^N})|\mathcal{H}_{t_k^N}]] \\ &= \sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|]\sigma_{\mathbf{a}\mathbf{a}^T}(t_{k+1}^N-t_k^N)^2\to 0,\end{aligned}$$

as well as

$$\begin{aligned}\mathbb{E}[|\sum_{k=0}^{N-1}\xi_k^N(t_{k+1}^N-t_k^N)(B^{\mathbf{a}}_{t_{k+1}^N}-B^{\mathbf{a}}_{t_k^N})|] &\le \sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|](t_{k+1}^N-t_k^N)\mathbb{E}[|B^{\mathbf{a}}_{t_{k+1}^N}-B^{\mathbf{a}}_{t_k^N}|] \\ &= \sqrt{\frac{2\sigma_{\mathbf{a}\mathbf{a}^T}}{\pi}}\sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|](t_{k+1}^N-t_k^N)^{3/2}\to 0\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}[|\sum_{k=0}^{N-1}\xi_k^N(\langle B^{\mathbf{a}}\rangle_{t_{k+1}^N}-\langle B^{\mathbf{a}}\rangle_{t_k^N})(B^{\bar{\mathbf{a}}}_{t_{k+1}^N}-B^{\bar{\mathbf{a}}}_{t_k^N})|] \\
&\leq \sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|]\mathbb{E}[(\langle B^{\mathbf{a}}\rangle_{t_{k+1}^N}-\langle B^{\mathbf{a}}\rangle_{t_k^N})|B^{\bar{\mathbf{a}}}_{t_{k+1}^N}-B^{\bar{\mathbf{a}}}_{t_k^N}|] \\
&\leq \sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|]\mathbb{E}[(\langle B^{\mathbf{a}}\rangle_{t_{k+1}^N}-\langle B^{\mathbf{a}}\rangle_{t_k^N})^2]^{1/2}\mathbb{E}[|B^{\bar{\mathbf{a}}}_{t_{k+1}^N}-B^{\bar{\mathbf{a}}}_{t_k^N}|^2]^{1/2} \\
&= \sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|]\sigma^{1/2}_{\mathbf{a}\mathbf{a}^T}\sigma^{1/2}_{\bar{\mathbf{a}}\bar{\mathbf{a}}^T}(t_{k+1}^N-t_k^N)^{3/2}\to 0.
\end{aligned}$$

We now can claim our G–Itô's formula. Consider

$$X_t^\nu = X_0^\nu + \int_0^t \alpha_s^\nu ds + \int_0^t \eta_s^{\nu ij} d\left\langle B^i, B^j\right\rangle_s + \int_0^t \beta_s^{\nu j} dB_s^j$$

Proposition 46 *Let α^ν, $\beta^{\nu j}$ and $\eta^{\nu ij}$, $\nu=1,\cdots,n$, $i,j=1,\cdots,d$ be bounded processes of $M_G^2(0,T)$. Then for each $t\geq 0$ and in $L_G^2(\mathcal{H}_t)$ we have*

$$\begin{aligned}
\Phi(X_t)-\Phi(X_s) = &\int_s^t \partial_{x^\nu}\Phi(X_u)\beta_u^{\nu j}dB_u^j + \int_s^t \partial_{x_\nu}\Phi(X_u)\alpha_u^\nu du \\
&+ \int_s^t [\partial_{x^\nu}\Phi(X_u)\eta_u^{\nu ij} + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u)\beta_u^{\nu i}\beta_u^{\nu j}]d\left\langle B^i, B^j\right\rangle_u
\end{aligned} \tag{27}$$

Proof. We first consider the case where α, η and β are step processes of the form

$$\eta_t(\omega) = \sum_{k=0}^{N-1}\xi_k(\omega)\mathbf{I}_{[t_k,t_{k+1})}(t).$$

From the above Lemma, it is clear that (27) holds true. Now let

$$X_t^{\nu,N} = X_0^\nu + \int_0^t \alpha_s^{\nu,N} ds + \int_0^t \eta_s^{\nu ij,N} d\left\langle B^i, B^j\right\rangle_s + \int_0^t \beta_s^{\nu j,N} dB_s^j$$

where α^N, η^N and β^N are uniformly bounded step processes that converge to α, η and β in $M_G^2(0,T)$ as $N\to\infty$. From Lemma 44

$$\begin{aligned}
\Phi(X_t^N)-\Phi(X_0) = &\int_0^t \partial_{x^\nu}\Phi(X_u^N)\beta_u^{\nu j,N}dB_u^j + \int_0^t \partial_{x_\nu}\Phi(X_u^N)\alpha_u^{\nu,N} du \\
&+ \int_0^t [\partial_{x^\nu}\Phi(X_u^N)\eta_u^{\nu ij,N} + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u^N)\beta_u^{\mu i,N}\beta_u^{\nu j,N}]d\left\langle B^i, B^j\right\rangle_u
\end{aligned} \tag{28}$$

Since

$$\mathbb{E}[|X_t^{N,\mu} - X_t^\mu|^2]$$
$$\leq C\int_0^T \{\mathbb{E}[(\alpha_s^{\mu,N} - \alpha_s^\mu)^2] + \mathbb{E}[|\eta_s^{\mu,N} - \eta_s^\mu|^2] + \mathbb{E}[(\beta_s^{\mu,N} - \beta_s^\mu)^2]\}ds$$

We then can prove that, in $M_G^2(0,T)$,

$$\begin{aligned}
\partial_{x^\nu}\Phi(X_\cdot^N)\eta_\cdot^{\nu ij,N} &\to \partial_{x^\nu}\Phi(X_\cdot)\eta_\cdot^{\nu ij} \\
\partial^2_{x^\mu x^\nu}\Phi(X_\cdot^N)\beta_\cdot^{\mu i,N}\beta_\cdot^{\nu j,N} &\to \partial^2_{x^\mu x^\nu}\Phi(X_\cdot)\beta_\cdot^{\mu i}\beta_\cdot^{\nu j} \\
\partial_{x_\nu}\Phi(X_\cdot^N)\alpha_\cdot^{\nu,N} &\to \partial_{x_\nu}\Phi(X_\cdot)\alpha_\cdot^\nu \\
\partial_{x^\nu}\Phi(X_\cdot^N)\beta_\cdot^{\nu j,N} &\to \partial_{x^\nu}\Phi(X_\cdot)\beta_\cdot^{\nu j}
\end{aligned}$$

We then can pass limit in both sides of (28) and thus prove (27). ■

6 G–martingales, G–convexity and Jensen's inequality

6.1 The notion of G–martingales

We now give the notion of G–martingales:

Definition 47 *A process $(M_t)_{t\geq 0}$ is called a G–**martingale** (resp. G–**supermartingale**, G–**submartingale**) if for each $0 \leq s \leq t < \infty$, we have $M_t \in L_G^1(\mathcal{H}_t)$ and*

$$\mathbb{E}[M_t|\mathcal{H}_s] = M_s, \quad (\text{resp. } \leq M_s, \ \geq M_s).$$

It is clear that, for a fixed $X \in L_G^1(\mathcal{H})$, $\mathbb{E}[X|\mathcal{H}_t]_{t\geq 0}$ is a G–martingale. In general, how to characterize a G–martingale or a G–supermartingale is still a very interesting problem. But the following example gives an important characterization:

Example 48 *Let $M_0 \in \mathbb{R}$, $\phi = (\phi^i)_{i=1}^d \in M_G^2(0,T;\mathbb{R}^d)$ and $\eta = (\eta^{ij})_{i,j=1}^d \in M_G^2(0,T;\mathbb{S}_d)$ be given and let*

$$M_t = M_0 + \int_0^t \phi_u^i dB_s^j + \int_0^t \eta_u^{ij} d\left\langle B^i, B^j\right\rangle_u - \int_0^t 2G(\eta_u)du, \ t \in [0,T].$$

Then M is a G–martingale on $[0,T]$. To prove this it suffices to prove the case $\eta \in M_G^{2,0}(0,T;\mathbb{S}_d)$, i.e.,

$$\eta_t = \sum_{k=0}^{N-1} \eta_{t_k} I_{[t_k.t_{k+1})}(t).$$

We have, for $s \in [t_{N-1}, t_N]$,

$$\begin{aligned}
\mathbb{E}[M_t|\mathcal{H}_s] &= M_s + \mathbb{E}[\eta^{ij}_{t_{N-1}}(\langle B^i, B^j\rangle_t - \langle B^i, B^j\rangle_s) - 2G(\eta_{t_{N-1}})(t-s)|\mathcal{H}_s] \\
&= M_s + \mathbb{E}[\eta^{ij}_{t_{N-1}}(B^i_t - B^i_s)(B^j_t - B^j_s)|\mathcal{H}_s] - 2G(\eta_{t_{N-1}})(t-s) \\
&= M_s.
\end{aligned}$$

In the last step, we apply the relation (11). We then can repeat this procedure, step by step backwardly, to prove the case $s \in [0, t_{N-1}]$.

Remark 49 *It is worth to mention that for a G–martingale, in general, $-M$ is not a G–martingale. But in the above example, when $\eta \equiv 0$, then $-M$ is still a G–martingale. This makes an essential difference of the dB part and the $d\langle B\rangle$ part of a G–martingale.*

6.2 G–convexity and Jensen's inequality for G–expectation

A very interesting question is whether the well–known Jensen's inequality still holds for G–expectation. In the framework of g–expectation, this problem was investigated in [3] in which a counterexample is given to indicate that, even for a linear function which is obviously convex, Jensen's inequality for g-expectation generally does not hold. Stimulated by this example, [30] proved that Jensen's inequality holds for any convex function under a g–expectation if and only if the corresponding generating function $g = g(t, z)$ is super-homogeneous in z. Here we will discuss this problem in a quite different point of view. We will define a new notion of convexity:

Definition 50 *A C^2-function $h : \mathbb{R} \longmapsto \mathbb{R}$ is called G–**convex** if the following condition holds for each $(y, z, A) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d$:*

$$G(h'(y)A + h''(y)zz^T) - h'(y)G(A) \geq 0, \tag{29}$$

where h' and h'' denote the first and the second derivatives of h.

It is clear that in the special situation where $G(D^2u) = \frac{1}{2}\Delta u$, the corresponding G-convex function becomes a classical convex function.

Lemma 51 *The following two conditions are equivalent:*
(i) *the function h is G–convex.*
(ii) *The following Jensen inequality holds: for each $T \geq 0$,*

$$\mathbb{E}[h(\phi(B_T))] \geq h(\mathbb{E}[\phi(B_T)]), \tag{30}$$

for each C^2–function ϕ such that $h(\phi(B_T))$, $\phi(B_T) \in L^1_G(\mathcal{H}_T)$.

Proof. (i) $\Longrightarrow$(ii) By the definition $u(t, x) := P^G_t[\phi](x) = \mathbb{E}[\phi(x + B_t)]$ solves the nonlinear heat equation (1). Here we only consider the case where u is a

$C^{1,2}$-function. Otherwise we can use the language of viscosity solution as we did in the proof of Lemma 6. By simple calculation, we have

$$\partial_t h(u(t,x)) = h'(u)\partial_t u = h'(u(t,x))G(D^2u(t,x)),$$

or

$$\partial_t h(u(t,x)) - G(D^2h(u(t,x))) - f(t,x) = 0,\ h(u(0,x)) = h(\phi(x)),$$

where we denote

$$f(t,x) = h'(u(t,x))G(D^2u(t,x)) - G(D^2h(u(t,x))).$$

Since h is G–convex, it follows that $f \leq 0$ and thus $h(u)$ is a G-subsolution. It follows from the maximum principle that $h(P_t^G(\phi)(x)) \leq P_t^G(h(\phi))(x)$. In particular (30) holds. Thus we have (ii).
(ii) $\Longrightarrow$(i): For a fixed $(y,z,A) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{S}_d$, we set $\phi(x) := y + \langle x,z\rangle + \frac{1}{2}\langle Ax,x\rangle$. By the definition of P_t^G we have $\partial_t(P_t^G(\phi)(x))|_{t=0} = G(D^2\phi)(x)$. By (ii) we have

$$h(P_t^G(\phi)(x)) \leq P_t^G(h(\phi))(x).$$

Thus, for $t>0$,

$$\frac{1}{t}[h(P_t^G(\phi)(x)) - h(\phi(x))] \leq \frac{1}{t}[P_t^G(h(\phi))(x) - h(\phi(x))]$$

We then let t tend to 0:

$$h'(\phi(x))G(D^2\phi(x)) \leq G(D_{xx}^2 h(\phi(x))).$$

Since $D_x\phi(x) = z + Ax$ and $D_{xx}^2\phi(x) = A$. We then set $x=0$ and obtain (29). ■

Proposition 52 *The following two conditions are equivalent:*
(i) *the function h is G–convex.*
(ii) *The following Jensen inequality holds:*

$$\mathbb{E}[h(X)|\mathcal{H}_t] \geq h(\mathbb{E}[X|\mathcal{H}_t]), \quad t \geq 0, \tag{31}$$

for each $X \in L_G^1(\mathcal{H})$ such that $h(X) \in L_G^1(\mathcal{H})$.

Proof. The part (ii) $\Longrightarrow$(i) is already provided by the above Lemma. We can also apply this lemma to prove (31) for the case $X \in L_{ip}^0(\mathcal{H})$ of the form $X = \phi(B_{t_1},\cdots,B_{t_m} - B_{t_{m-1}})$ by using the procedure of the definition of $\mathbb{E}[\cdot]$ and $\mathbb{E}[\cdot|\mathcal{H}_t]$ given in Definition 13. We then can extend this Jensen's inequality, under the norm $\|\cdot\| = \mathbb{E}[|\cdot|]$, to the general situation. ■

Remark 53 *The above notion of G–convexity can be also applied to the case where the nonlinear heat equation (1) has a more general form:*

$$\frac{\partial u}{\partial t} - G(u,\nabla u, D^2u) = 0, \quad u(0,x) = \psi(x) \tag{32}$$

*(see Examples 4.3, 4.4 and 4.5 in [44]). In this case a C^2-function $h : \mathbb{R} \longmapsto \mathbb{R}$ is called to be G–**convex** if the following condition holds for each $(y, z, A) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d$:*

$$G(y, h'(y)z, h'(y)A + h''(y)zz^T) - h'(y)G(y, z, A) \geq 0.$$

We don't need the subadditivity and/or positive homogeneity of $G(y, z, A)$. A particularly interesting situation is the case of g–expectation for a given generating function $g = g(y, z)$, $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, in this case we have the following g–convexity:

$$\frac{1}{2}h''(y)|z|^2 + g(h(y), h'(y)z) - h'(y)g(y, z) \geq 0. \tag{33}$$

We will discuss this situation elsewhere.

Example 54 *Let h be a G–convex function and let $X \in L^1_G(\mathcal{H})$ be such that $h(X) \in L^1_G(\mathcal{H})$, then $Y_t = h(\mathbb{E}[X|\mathcal{H}_t])$, $t \geq 0$, is a G–submartingale: for each $s \leq t$,*

$$\mathbb{E}[Y_t|\mathcal{H}_s] = \mathbb{E}[h(\mathbb{E}[X|\mathcal{F}_t])|\mathcal{F}_s] \geq h(\mathbb{E}[X|\mathcal{F}_s]) = Y_s.$$

7 Stochastic differential equations

We consider the following SDE driven by G-Brownian motion.

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t h_{ij}(X_s)d\left\langle B^i, B^j \right\rangle_s + \int_0^t \sigma_j(X_s)dB^j_s, \ t \in [0, T]. \tag{34}$$

where the initial condition $X_0 \in \mathbb{R}^n$ is given and

$$b, h_{ij}, \sigma_j : \mathbb{R}^n \mapsto \mathbb{R}^n$$

are given Lipschitz functions, i.e., $|\phi(x) - \phi(x')| \leq K|x - x'|$, for each x, $x' \in \mathbb{R}^n$, $\phi = b$, η_{ij} and σ_j. Here the horizon $[0, T]$ can be arbitrarily large. The solution is a process $X \in M^2_G(0, T; \mathbb{R}^n)$ satisfying the above SDE. We first introduce the following mapping on a fixed interval $[0, T]$:

$$\Lambda_{\cdot}(Y) :=: Y \in M^2_G(0, T; \mathbb{R}^n) \longmapsto M^2_G(0, T; \mathbb{R}^n)$$

by setting $\Lambda_t = X_t$, $t \in [0, T]$, with

$$\Lambda_t(Y) = X_0 + X_0 + \int_0^t b(Y_s)ds + \int_0^t h_{ij}(Y_s)d\left\langle B^i, B^j \right\rangle_s + \int_0^t \sigma_j(Y_s)dB^j_s.$$

We immediately have

Lemma 55 *For each $Y, Y' \in M^2_G(0, T; \mathbb{R}^n)$, we have the following estimate:*

$$\mathbb{E}[|\Lambda_t(Y) - \Lambda_t(Y')|^2] \leq C \int_0^t \mathbb{E}[|Y_s - Y'_s|^2]ds, \ t \in [0, T],$$

where the constant C depends only on K, Γ and the dimension n.

Proof. This is a direct consequence of the inequalities (14), (16) and (23). ■

We now prove that SDE (34) has a unique solution. By multiplying e^{-2Ct} on both sides of the above inequality and then integrate them on $[0,T]$. It follows that

$$\begin{aligned}\int_0^T \mathbb{E}[|\Lambda_t(Y)-\Lambda_t(Y')|^2]e^{-2Ct}dt &\leq C\int_0^T e^{-2Ct}\int_0^t \mathbb{E}[|Y_s-Y_s'|^2]dsdt \\ &= C\int_0^T\int_s^T e^{-2Ct}dt\mathbb{E}[|Y_s-Y_s'|^2]ds \\ &= (2C)^{-1}C\int_0^T (e^{-2Cs}-e^{-2CT})\mathbb{E}[|Y_s-Y_s'|^2]ds.\end{aligned}$$

We then have

$$\int_0^T \mathbb{E}[|\Lambda_t(Y)-\Lambda_t(Y')|^2]e^{-2Ct}dt \leq \frac{1}{2}\int_0^T \mathbb{E}[|Y_t-Y_t'|^2]e^{-2Ct}dt.$$

We observe that the following two norms are equivalent in $M_G^2(0,T;\mathbb{R}^n)$

$$\int_0^T \mathbb{E}[|Y_t|^2]dt \sim \int_0^T \mathbb{E}[|Y_t|^2]e^{-2Ct}dt.$$

From this estimate we can obtain that $\Lambda(Y)$ is a contract mapping. Consequently, we have

Theorem 56 *There exists a unique solution of $X \in M_G^2(0,T;\mathbb{R}^n)$ of the stochastic differential equation (34).*

8 Appendix: Some inequalities in $L_G^p(\mathcal{H})$

For $r>0$, $1<p,q<\infty$, such that $\frac{1}{p}+\frac{1}{q}=1$, we have

$$|a+b|^r \leq \max\{1,2^{r-1}\}(|a|^r+|b|^r), \quad \forall a,b\in\mathbb{R} \tag{35}$$

$$|ab| \leq \frac{|a|^p}{p}+\frac{|b|^q}{q}. \tag{36}$$

Proposition 57

$$\mathbb{E}[|X+Y|^r] \leq C_r(\mathbb{E}[|X|^r]+\mathbb{E}[|Y|^r]), \tag{37}$$

$$\mathbb{E}[|XY|] \leq \mathbb{E}[|X|^p]^{1/p}\cdot\mathbb{E}[|Y|^q]^{1/q}, \tag{38}$$

$$\mathbb{E}[|X+Y|^p]^{1/p} \leq \mathbb{E}[|X|^p]^{1/p}+\mathbb{E}[|Y|^p]^{1/p} \tag{39}$$

In particular, for $1\leq p<p'$, we have $\mathbb{E}[|X|^p]^{1/p}\leq\mathbb{E}[|X|^{p'}]^{1/p'}$.

Proof. (37) follows from (35). We set

$$\xi = \frac{X}{\mathbb{E}[|X|^p]^{1/p}}, \quad \eta = \frac{Y}{\mathbb{E}[|Y|^q]^{1/q}}.$$

By (36) we have

$$\begin{aligned}\mathbb{E}[|\xi\eta|] &\leq \mathbb{E}[\frac{|\xi|^p}{p} + \frac{|\eta|^q}{q}] \leq \mathbb{E}[\frac{|\xi|^p}{p}] + \mathbb{E}[\frac{|\eta|^q}{q}] \\ &= \frac{1}{p} + \frac{1}{q} = 1.\end{aligned}$$

Thus (38) follows.

$$\begin{aligned}\mathbb{E}[|X+Y|^p] &= \mathbb{E}[|X+Y| \cdot |X+Y|^{p-1}] \\ &\leq \mathbb{E}[|X| \cdot |X+Y|^{p-1}] + \mathbb{E}[|Y| \cdot |X+Y|^{p-1}] \\ &\leq \mathbb{E}[|X|^p]^{1/p} \cdot \mathbb{E}[|X+Y|^{(p-1)q}]^{1/q} \\ &+ \mathbb{E}[|Y|^p]^{1/p} \cdot \mathbb{E}[|X+Y|^{(p-1)q}]^{1/q}\end{aligned}$$

We observe that $(p-1)q = p$. Thus we have (39). ∎

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